1. [2360/102324 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) If $\mathbf{A}$ is 2×3 and $\mathbf{B}$ is 3×4, then $(\mathbf{AB})^{\mathrm{T}}$ is 2×4.

   (b) Given a vector space $\mathbb{V}$, any set of vectors in $\mathbb{V}$ containing $\vec{\mathbf{0}}$ is linearly dependent.

   (c) If $\mathbf{G}$ is an $n \times n$ matrix and m is a positive integer, then $|\mathbf{G}^m| = |\mathbf{G}|^m$.

   (d) Cramer's Rule can be used to solve $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ if $\mathbf{A}$ is 5×5 and rank $\mathbf{A} = 4$

   (e) If $\mathbf{A}$ is a 10×10 matrix with $|\mathbf{A}| = 2$ and $\mathbf{B}$ is the matrix obtained by multiplying row 4 of $\mathbf{A}$ by 4, then $|\mathbf{B}| = 8$.

SOLUTION:

(a) **FALSE** $(\mathbf{AB})^{\mathrm{T}}$ is 4×2.

(b) **TRUE** $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{0}} + \cdots + c_n\vec{\mathbf{v}}_n = \vec{\mathbf{0}}$ has nontrivial solutions (c_3 can be any real number).

(c) **TRUE** $|\mathbf{G}^m| = \underbrace{|(\mathbf{G})(\mathbf{G})\cdots(\mathbf{G})|}_{m \text{ factors}} = \underbrace{|\mathbf{G}||\mathbf{G}|\cdots|\mathbf{G}|}_{m \text{ factors}} = |\mathbf{G}|^m$

(d) **FALSE** Since rank $\mathbf{A} < 5$, $|\mathbf{A}| = 0$ and Cramer's Rule cannot be used.

(e) **TRUE** Multiplying a row of a matrix by a nonzero constant multiplies the determinant by that nonzero constant.

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2. [2360/102324 (14 pts)] Use Gauss-Jordan row reduction to find the inverse of an appropriate matrix and use that inverse matrix to find the solution of

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 + x_2 + x_3 &= -2 \\ x_2 + x_3 &= -3 \end{aligned}$$

SOLUTION:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$

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3. [2360/102324 (19 pts)] The following problems are not related.

   (a) (7 pts) What does the Wronskian tell you about the linear independence of the functions $\{x, 1+x, 2+3x\}$ on the real line?

   (b) (12 pts) Determine which of the following sets of vectors form a basis for the given vector space, $\mathbb{V}$. Justify your answers.

      i. (4 pts) $\mathbb{V} = \mathbb{R}^2$; $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\}$

      ii. (4 pts) $\mathbb{V} = \mathbb{R}^3$; $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \right\}$

      iii. (4 pts) $\mathbb{V} = \mathbb{P}_3$; any set of 3 linearly independent third degree polynomials

SOLUTION:

(a)

$$W(x) = \begin{vmatrix} x & 1+x & 2+3x \\ 1 & 1 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Since $W(x) \equiv 0$, we can conclude nothing about the linear independence of the functions.

(b) i. No. Justification (1): The set contains 3 vectors in a vector space of dimension 2 (too many).

Justification (2): The vectors are linearly dependent: $3\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

ii. Yes. There are three linearly independent vectors in a vector space of dimension 3 so they span the vector space.

$$\begin{vmatrix} 2 & 0 & 4 \\ 0 & -2 & 1 \\ 1 & 3 & 1 \end{vmatrix} = 2(-1)^{1+1}\begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix} + 4(-1)^{1+3}\begin{vmatrix} 0 & -2 \\ 1 & 3 \end{vmatrix} = 2(-5) + 4(2) = -2 \neq 0$$

iii. No. There are only 3 vectors in a vector space of dimension 4 (too few); they cannot span the space

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4. [2360/102324 (13 pts)] Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 8 & 10 \\ -3 & -7 & -5 \end{bmatrix}$.

(a) (10 pts) For what value(s) of k is $\vec{\mathbf{b}} = \begin{bmatrix} -1 & k & 1 \end{bmatrix}^{\mathrm{T}} \in \text{Col } \mathbf{A}$?

(b) (3 pts) For what value(s) of k is the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ inconsistent?

SOLUTION:

(a)

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 2 & 8 & 10 & k \\ -3 & -7 & -5 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & 4 & 8 & k+2 \\ 0 & -1 & -2 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 4 & 8 & k+2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & k-6 \end{array}\right]$$

$\vec{\mathbf{b}}$ will be in the column space of $\mathbf{A}$ if $k = 6$.

(b) From part (a), the system will be inconsistent if $k \neq 6$.

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5. [2360/102324 (20 pts)] Consider the following augmented matrix of the linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$:

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & -1 \\ -2 & -5 & -3 & -1 \\ 5 & 16 & 4 & -8 \\ 0 & -1 & 1 & 3 \end{array}\right]$$

(a) (2 pts) Is the system overdetermined, underdetermined or neither?

(b) (10 pts) Find the general solution, that is, solve the system.

(c) (8 pts) Find a basis for the solution space of the associated homogeneous system and find the dimension of that solution space. Your work from part (b) may prove beneficial.

SOLUTION:

(a) Since there are more equations (4) than variables (3), the system is overdetermined.

(b)

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & -1 \\ -2 & -5 & -3 & -1 \\ 5 & 16 & 4 & -8 \\ 0 & -1 & 1 & 3 \end{array}\right] \xrightarrow[R_3^*=-5R_1+R3]{R_2^*=2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 3 & 1 & -1 \\ 0 & 1 & -1 & -3 \\ 0 & 1 & -1 & -3 \\ 0 & -1 & 1 & 3 \end{array}\right] \xrightarrow[R_4^*=R_2+R_4]{R_1^*=-3R_2+R_1,\ R_3^*=-1R_2+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 8 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 is a free variable, which we can set to $t \in \mathbb{R}$. Then $x_1 = 8 - 4t, x_2 = -3 + t$ and $\vec{\mathbf{x}} = \begin{bmatrix} 8-4t \\ -3+t \\ t \end{bmatrix}$, $t \in \mathbb{R}$ is the general solution.

(c) The solution to the associated homogeneous system is $\vec{\mathbf{x}} = \begin{bmatrix} -4t \\ t \\ t \end{bmatrix}$, $t \in \mathbb{R} = \text{span}\left\{\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}\right\}$. A basis is thus $\left\{\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}\right\}$ having a dimension of one.

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6. [2360/102324 (12 pts)] Let $\mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

(a) (6 pts) Find the eigenvalues of $\mathbf{B}$ and state their algebraic multiplicity.

(b) (6 pts) Find a basis for the eigenspace associated with the largest eigenvalue you found in part (a). What is the geometric multiplicity of this largest eigenvalue?

SOLUTION:

(a)

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)(-1)^{3+3}\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left[(1-\lambda)^2 - 1\right] = (1-\lambda)(1 - 2\lambda + \lambda^2 - 1) = (1-\lambda)(\lambda)(-2+\lambda) = 0$$

The eigenvalues are $\lambda = 0, 1, 2$, each having an algebraic multiplicity of 1.

(b) We need to solve the system $(\mathbf{A} - 2\mathbf{I})\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

A basis for the eigenspace is $\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$. The geometric multiplicity of $\lambda = 2$ is 1, the dimension of its eigenspace.

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7. [2360/102324 (12 pts)] Determine if the following subsets, $\mathbb{W}$, are subspaces of the given vector space, $\mathbb{V}$. Justify your answer.

(a) (6 pts) $\mathbb{V} = \mathbb{R}^2$, $\mathbb{W}$ is the set of vectors of the form $\begin{bmatrix} a \\ a^2 \end{bmatrix}$ where a is a real number.

(b) (6 pts) $\mathbb{V} = \mathbb{M}_{22}$, $\mathbb{W}$ is the set of 2×2 matrices for which the entries in each column sum to zero.

SOLUTION:

(a) No. Let $\vec{\mathbf{u}} = \begin{bmatrix} a \\ a^2 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} b \\ b^2 \end{bmatrix} \in \mathbb{W}$ and $p, q \in \mathbb{R}$. Then

$$p\vec{\mathbf{u}} + q\vec{\mathbf{v}} = p\begin{bmatrix} a \\ a^2 \end{bmatrix} + q\begin{bmatrix} b \\ b^2 \end{bmatrix} = \begin{bmatrix} pa+qb \\ pa^2+qb^2 \end{bmatrix} \notin \mathbb{W} \quad \text{since} \begin{bmatrix} pa+qb \\ pa^2+qb^2 \end{bmatrix} \neq \begin{bmatrix} pa+qb \\ (pa+qb)^2 \end{bmatrix}$$

$\mathbb{W}$ is not closed. You can also find specific counterexamples showing the failure of either of the closure properties individually.

(b) Yes. $\vec{\mathbf{u}} = \begin{bmatrix} a & b \\ -a & -b \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} c & d \\ -c & -d \end{bmatrix} \in \mathbb{W}$ and $p, q \in \mathbb{R}$. Then

$$p\vec{\mathbf{u}} + q\vec{\mathbf{v}} = p\begin{bmatrix} a & b \\ -a & -b \end{bmatrix} + q\begin{bmatrix} c & d \\ -c & -d \end{bmatrix} = \begin{bmatrix} pa+qc & pb+qd \\ -(pa+qc) & -(pb+qd) \end{bmatrix} \in \mathbb{W}$$

since each column sums to 0.

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