1. [2360/121923 (24 pts)] Write the word **TRUE** or **FALSE** as appropriate. Write your answers in a single column separate from any work you do. No work need be shown. No partial credit given.

   (a) The set $\mathbb{W}$ consisting of vectors of the form $\vec{\mathbf{x}} = [a \quad b \quad 0 \quad a-b]^{\mathrm{T}}$ is a subspace of $\mathbb{R}^4$.

   (b) $y = 4$ is the only equilibrium solution of $y' = t(y-4)^2$.

   (c) If the Wronskian of two arbitrary functions is identically zero on the real line, then the two functions must always be linearly dependent on $\mathbb{R}$.

   (d) If $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is consistent, then $\vec{\mathbf{b}} \in \text{Col } \mathbf{A}$.

   (e) If $\vec{\mathbf{x}}$ is an $n \times 1$ matrix and $\mathbf{A}$ is an $n \times n$ matrix, then $\vec{\mathbf{x}}^{\mathrm{T}}\mathbf{A}\vec{\mathbf{x}}$ is an $n \times 1$ matrix.

   (f) The system $\begin{cases} x' = x(3-x-2y) \\ y' = y(2-y-x) \end{cases}$ has an equilibrium solution at the origin and a v nullcline of $y = 2 - x$.

   (g) Every first order linear homogeneous differential equation is separable.

   (h) If $\mathbf{A}$ is an $n \times n$ matrix with two eigenvalues equal to 0, then the columns of $\mathbf{A}$ are linearly dependent and the solution to $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$, where $\vec{\mathbf{b}}$ is an $n \times 1$ matrix, is $\vec{\mathbf{x}} = \mathbf{A}^{-1}\vec{\mathbf{b}}$.

SOLUTION:

(a) **TRUE** Let $\vec{\mathbf{x}} = \begin{bmatrix} a \\ b \\ 0 \\ a-b \end{bmatrix}$ and $\vec{\mathbf{y}} = \begin{bmatrix} c \\ d \\ 0 \\ c-d \end{bmatrix}$ be in $\mathbb{W}$ and $p, q \in \mathbb{R}$. Then

$$p\vec{\mathbf{x}} + q\vec{\mathbf{y}} = p\begin{bmatrix} a \\ b \\ 0 \\ a-b \end{bmatrix} + q\begin{bmatrix} c \\ d \\ 0 \\ c-d \end{bmatrix} = \begin{bmatrix} pa+qc \\ pb+qd \\ 0 \\ p(a-b)+q(c-d) \end{bmatrix} = \begin{bmatrix} pa+qc \\ pb+qd \\ 0 \\ (pa+qc)-(pb+qd) \end{bmatrix} \in \mathbb{W}$$

showing that $\mathbb{W}$ is closed and thus is a subspace.

(b) **TRUE** Equilibrium solutions must be constants and cannot contain the independent variable (t here).

(c) **FALSE** Vanishing Wronskians imply linear dependence only if the functions are solutions to a second order linear homogeneous differential equation. If not, $W \equiv 0$ implies nothing.

(d) **TRUE** For example, consider the consistent (meaning it has a solution) system

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \text{ which is equivalent to } x_1\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_2\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + x_3\begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} + x_4\begin{bmatrix} a_{14} \\ a_{24} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

showing that $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ is in the span of the columns of $\mathbf{A}$, that is, $\vec{\mathbf{b}} \in \text{Col } \mathbf{A}$.

(e) **FALSE** $(1 \times n)(n \times n)(n \times 1)$ yields a 1×1 matrix.

(f) **FALSE** The origin is an equilibrium solution but $y = 2 - x$ is an h nullcline.

(g) **TRUE** $\dfrac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = 0$ can be written as $\dfrac{\mathrm{d}y}{y} = -p(t)\,\mathrm{d}t$.

(h) **FALSE** Matrices having 0 as an eigenvalue are not invertible.

■

2. [2360/121923 (18 pts)] Let $f(t) = t\,\text{step}(t) - t\,\text{step}(t-1) - 3\,\text{step}(t-2)$.

   (a) (5 pts) Write $f(t)$ as a piecewise defined function.

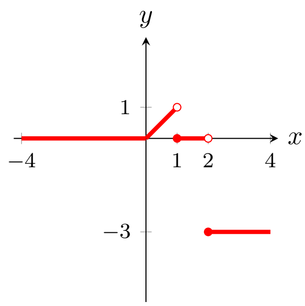
   (b) (5 pts) Make a well-labeled graph of $f(t)$ on the interval $[-4, 4]$.

   (c) (8 pts) Find $\mathscr{L}\{f(t)\}$.

SOLUTION:

(a) $f(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t < 1 \\ 0 & 1 \le t < 2 \\ -3 & 2 \le t \end{cases}$

(b) Graph of $f(t)$



(c)

$$\begin{aligned}\mathscr{L}\{t\,\text{step}(t) - t\,\text{step}(t-1) - 3\,\text{step}(t-2)\} &= \mathscr{L}\{t\,\text{step}(t)\} - \mathscr{L}\{t\,\text{step}(t-1)\} - 3\mathscr{L}\{\text{step}(t-2)\} \\ &= e^{-0s}\mathscr{L}\{t\} - e^{-s}\mathscr{L}\{t+1\} - 3e^{-2s}\mathscr{L}\{1\} \\ &= \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) - \frac{3e^{-2s}}{s}\end{aligned}$$

Alternatively,

$$\begin{aligned}\mathscr{L}\{t\,\text{step}(t)\} - \mathscr{L}\{t\,\text{step}(t-1)\} - 3\mathscr{L}\{\text{step}(t-2)\} &= -\frac{\mathrm{d}}{\mathrm{d}s}\mathscr{L}\{\text{step}\,(t)\} - (-1)\frac{\mathrm{d}}{\mathrm{d}s}\mathscr{L}\{\text{step}\,(t-1)\} - \frac{3e^{-2s}}{s} \\ &= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s}\right) + \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{e^{-s}}{s}\right) - \frac{3e^{-2s}}{s} = \frac{1}{s^2} + \frac{-se^{-s} - e^{-s}}{s^2} - \frac{3e^{-2s}}{s} \\ &= \frac{1}{s^2} - e^{-s}\left(\frac{1}{s} + \frac{1}{s^2}\right) - \frac{3e^{-2s}}{s}\end{aligned}$$

■

3. [2360/121923 (20 pts)] A mass-spring system at $t = 0$ features the 1-kg mass at rest at the equilibrium position. The restoring constant is 40 N/m and the system is hooked up so that the damping force is numerically equal to 4 times the instantaneous velocity. The oscillator is subjected to a driving force of $f(t) = 40 + \delta(t-2)$. Find the displacement, $x(t)$, of the mass for all $t > 0$.

SOLUTION:

We need to solve the initial value problem $\ddot{x} + 4\dot{x} + 40x = 40 + \delta(t-2),\ x(0) = \dot{x}(0) = 0$ and we use Laplace transforms to do it.

$$\mathscr{L}\left\{\ddot{x} + 4\dot{x} + 40x = 40 + \delta(t-2)\right\}$$

$$s^2X(s) - sx(0) - \dot{x}(0) + 4\left[sX(s) - x(0)\right] + 40X(s) = \frac{40}{s} + e^{-2s}$$

$$X(s) = \frac{40}{s\left(s^2+4s+40\right)} + \frac{e^{-2s}}{s^2+4s+40}$$

$$\frac{40}{s\left(s^2+4s+40\right)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+40}$$

$$40 = A\left(s^2+4s+40\right) + (Bs+C)s$$

$$s = 0 : 40 = A(40) \implies A = 1$$

$$\left.\begin{aligned} s = 1 : 40 = 45 + B + C &\implies B + C = -5 \\ s = -1 : 40 = 37 + B - C] &\implies B - C = 3 \end{aligned}\right\} \implies B = -1, C = -4$$

$$\frac{40}{s\left(s^2+4s+40\right)} = \frac{1}{s} - \frac{s+4}{(s+2)^2+36} = \frac{1}{s} - \frac{s+2+\frac{6}{3}}{(s+2)^2+6^2} = \frac{1}{s} - \frac{s+2}{(s+2)^2+6^2} - \frac{1}{3}\left[\frac{6}{(s+2)^2+6^2}\right]$$

$$X(s) = \frac{1}{s} - \frac{s+2}{(s+2)^2+6^2} - \frac{1}{3}\left[\frac{6}{(s+2)^2+6^2}\right] + e^{-2s}\left\{\frac{1}{6}\left[\frac{6}{(s+2)^2+6^2}\right]\right\}$$

$$x(t) = \mathscr{L}^{-1}\{X(s)\} = 1 - e^{-2t}\cos 6t - \frac{1}{3}e^{-2t}\sin 6t + \frac{1}{6}\left\{e^{-2(t-2)}\sin[6(t-2)]\right\}\text{step}(t-2)$$

$$= 1 - e^{-2t}\left(\cos 6t + \frac{1}{3}\sin 6t\right) + \frac{1}{6}\left[e^{4-2t}\sin(6t-12)\right]\text{step}(t-2)$$

■

4. [2360/121923 (15 pts)] Two 100-gallon tanks are completely full. Initially, tank 1 contains 5 pounds of dissolved sugar and tank 2 has 3 pounds of dissolved sugar in it. The contents in the tanks are well stirred. The flow rate into tank 1 is always 20 gallons per minute (gpm). For $0 \le t < 4$, fresh water enters tank 1. For $t \ge 4$, the water entering tank 1 contains t pounds of sugar per gallon. For all $t \ge 0$, solution from tank 1 enters tank 2 at 25 gpm; also, solution from tank 2 enters tank 1 at 5 gpm and leaves tank 2 at 20 gpm. Set up, but **do not solve**, an initial value problem whose solution will give the amount of sugar in each tank for all time. Write your final answer using matrices and vectors.

SOLUTION:

We use the general idea of rate of change of mass equals mass rate in minus mass rate out. Note that the unit *ppg* means pounds per gallon and that the volume of solution in each tank remains 100 gallons at all times. Let $x_1(t), x_2(t)$ be the amount of sugar (lb) in tank 1 and tank 2, respectively.

$$x_1' = (20\,\text{gpm})\left[t\,\text{step}(t-4)\,\text{ppg}\right] + (5\,\text{gpm})\left(\frac{x_2}{100}\,\text{ppg}\right) - (25\,\text{gpm})\left(\frac{x_1}{100}\,\text{ppg}\right) = -\frac{1}{4}x_1 + \frac{1}{20}x_2 + 20t\,\text{step}(t-4)$$

$$x_2' = (25\,\text{gpm})\left(\frac{x_1}{100}\,\text{ppg}\right) - (25\,\text{gpm})\left(\frac{x_2}{100}\,\text{ppg}\right) = \frac{1}{4}x_1 - \frac{1}{4}x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -\frac{1}{4} & \frac{1}{20} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 20t\,\text{step}(t-4) \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

■

5. [2360/121923 (12 pts) A certain object's temperature, $T(t)$, is governed by the differential equation $\dfrac{\mathrm{d}T}{\mathrm{d}t} = 2\left(te^{-2t} - T\right)$. If it's temperature when $t = 0$ is 1, find its temperature when $t = 1$.

SOLUTION:

This can be solved using the integrating factor method or variation of parameters (Euler-Lagrange Two-Stage Method) or Laplace transforms. We use the former here. Begin by rewriting the ODE as $T' + 2T = 2te^{-2t}$, showing that $p(t) = 2$, we have

$$\mu(t) = e^{\int 2\,\mathrm{d}t} = e^{2t}$$
$$e^{2t}\left(T' + 2T = 2te^{-2t}\right)$$
$$\left(e^{2t}T\right)' = 2t$$
$$e^{2t}T = \int \left(e^{2t}T\right)'\,\mathrm{d}t = \int 2t\,\mathrm{d}t = t^2 + C$$
$$T(t) = \left(t^2 + C\right)e^{-2t}$$
$$T(0) = \left(0^2 + C\right)e^0 = 1 \implies C = 1$$
$$T(t) = \left(t^2 + 1\right)e^{-2t}$$
$$T(1) = 2e^{-2}$$

Alternatively, using variation of parameters, the solution to the homogeneous problem is, via separation of variables, $T_h = ce^{-2t}$. Then $T_p = v(t)e^{-2t}$, which upon substituting into the nonhomogeneous equation, yields $v'(t) = 2t \implies v(t) = t^2$ so that $T_p = t^2e^{-2t}$. Application of the Nonhomogeneous Principle gives $T(t) = T_h(t) + T_p(t) = ce^{-2t} + t^2e^{-2t}$. With the initial condition we have $T(t) = \left(t^2 + 1\right)e^{-2t}$.

■

6. [2360/121923 (18 pts)] Let $\mathbf{A} = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$.

   (a) (4 pts) Use the definition of eigenvalues/eigenvectors to show that $\lambda = i$ and $\vec{\mathbf{v}} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$ are an eigenvalue/eigenvector pair of matrix $\mathbf{A}$. No credit for using determinants.

   (b) (14 pts) Solve the initial value problem $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}},\ \vec{\mathbf{x}}(0) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, writing your answer as a single vector.

SOLUTION:

(a) $\mathbf{A}\vec{\mathbf{v}} = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1-i \\ 1 \end{bmatrix} = \begin{bmatrix} 1+i \\ i \end{bmatrix} = i\begin{bmatrix} 1-i \\ 1 \end{bmatrix} = \lambda\vec{\mathbf{v}}$

(b) The eigenvalue is $\lambda = i$ so that $\alpha = 0$ and $\beta = 1$. The eigenvector is $\vec{\mathbf{v}} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \vec{\mathbf{p}} + i\vec{\mathbf{q}}$.

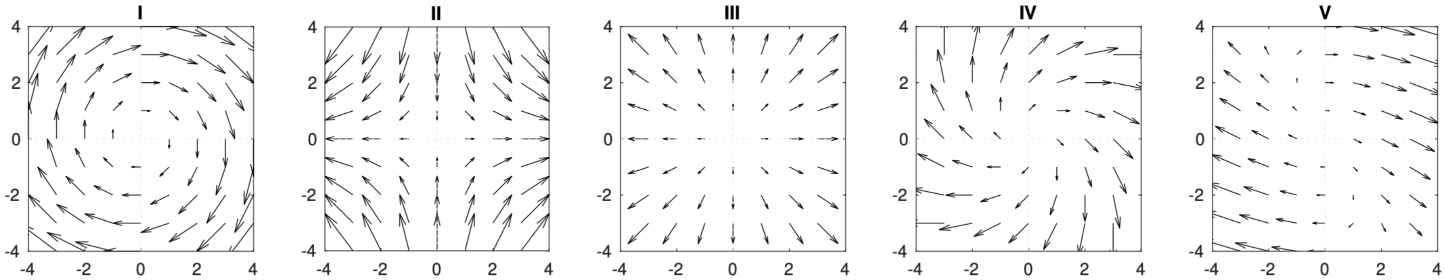
$$\vec{\mathbf{x}}(t) = c_1\left(\cos t\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin t\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) + c_2\left(\sin t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \cos t\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$$
$$\vec{\mathbf{x}}(0) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \implies c_1 = 2, c_2 = -3$$
$$\vec{\mathbf{x}}(t) = 2\begin{bmatrix} \cos t + \sin t \\ \cos t \end{bmatrix} - 3\begin{bmatrix} \sin t - \cos t \\ \sin t \end{bmatrix} = \begin{bmatrix} 5\cos t - \sin t \\ 2\cos t - 3\sin t \end{bmatrix}$$

■

7. [2360/121923 (10 pts)] Consider the system of differential equations $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$. Match the phase portrait to the appropriate system for the given matrices. Write your answers in a single column separate from any work you do. No work need be shown. No partial credit

given.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ (b) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (c) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (d) $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ (e) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$



SOLUTION:

(a) IV; $\text{Tr}\,\mathbf{A} = 2, |\mathbf{A}| = 2, (\text{Tr}\,\mathbf{A})^2 - 4|\mathbf{A}| = -4 < 0$; unstable spiral

(b) III; $\text{Tr}\,\mathbf{A} = 2, |\mathbf{A}| = 1, (\text{Tr}\,\mathbf{A})^2 - 4|\mathbf{A}| = 0$; unstable star node (two linearly independent eigenvectors)

(c) I; $\text{Tr}\,\mathbf{A} = 0, |\mathbf{A}| = 1, (\text{Tr}\,\mathbf{A})^2 - 4|\mathbf{A}| = -4 < 0$; neutrally stable center

(d) V; $\text{Tr}\,\mathbf{A} = 2, |\mathbf{A}| = 1, (\text{Tr}\,\mathbf{A})^2 - 4|\mathbf{A}| = 0$; unstable degenerate node (one eigenvector)

(e) II; $\text{Tr}\,\mathbf{A} = 0, |\mathbf{A}| = -1, (\text{Tr}\,\mathbf{A})^2 - 4|\mathbf{A}| = 4 > 0$; unstable saddle

8. [2360/121923 (33 pts)] Consider the matrix $\mathbf{A} = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix}$.

(a) (5 pts) Show that $|\mathbf{A}| = c^3 - 3c + 2$ by using the cofactor expansion method, expanding along the first row.

(b) (2 pts) Verify that $c^3 - 3c + 2 = (c+2)(c^2 - 2c + 1)$.

(c) (4 pts) Using the result in part (b), find the roots of $c^3 - 3c + 2 = 0$ and state the multiplicity of each.

(d) (12 pts) Using the information gathered in parts (a), (b), and (c), determine the number of solutions to the system $\mathbf{A}\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ if

i. $c = 1$ ii. $c \neq 1, -2$ iii. $c = -2$

(e) (5 pts) Find a basis for the solution space of $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ when $c = 1$. What is the dimension of the solution space for this case?

(f) (5 pts) Suppose the characteristic equation of a linear, homogeneous, constant coefficient differential equation is $r^3 - 3r + 2 = 0$. Use the information from parts (a), (b) and (c) to answer the following questions.

i. (3 pts) Find a basis for the solution space of the differential equation.

ii. (2 pts) Find the form of the particular solution you would use in the Method of Undetermined Coefficients if the differential equation was forced by the nonhomogeneous term $f(t) = e^{2t} + t^2 e^t$. **Do not** solve for the coefficients.

SOLUTION:

(a)

$$\begin{vmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{vmatrix} = c(-1)^{1+1} \begin{vmatrix} c & 1 \\ 1 & c \end{vmatrix} + 1(-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 1 & c \end{vmatrix} + 1(-1)^{1+3} \begin{vmatrix} 1 & c \\ 1 & 1 \end{vmatrix}$$

$$= c\left(c^2 - 1\right) - (c-1) + (1-c) = c^3 - c - c + 1 + 1 - c = c^3 - 3c + 2$$

(b) $(c+2)(c^2 - 2c + 1) = c^3 - 2c^2 + c + 2c^2 - 4c + 2 = c^3 - 3c + 2$

(c) We can now fully factor: $c^3 - 3c + 2 = (c+2)(c-1)^2 = 0$ giving the roots of -2 with multiplicity 1 and 1 with multiplicity 2.

(d) i. If $c = 1$, $|\mathbf{A}| = 0$, from which we can conclude nothing about the number of solutions to the system. Instead, use the RREF.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -2 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

implying that there are no solutions to the system.

ii. If $c \neq 1, -2$, $|\mathbf{A}| \neq 0$ implying that there is a single, unique solution to the system.

iii. If $c = -2$, $|\mathbf{A}| = 0$, and again we can conclude nothing about the number of solutions to the system, so use the RREF.

$$\left[\begin{array}{rrr|r} -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & -2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{rrr|r} 1 & 1 & -2 & -2 \\ 1 & -2 & 1 & 1 \\ -2 & 1 & 1 & 1 \end{array}\right] \xrightarrow[R_3^*=2R_1+R_3]{R_2^*=-1R_1+R_2} \left[\begin{array}{rrr|r} 1 & 1 & -2 & -2 \\ 0 & -3 & 3 & 3 \\ 0 & 3 & -3 & -3 \end{array}\right]$$

$$\xrightarrow[R_2^*=-\frac{1}{3}R_2]{R_3^*=1R_2+R_3} \left[\begin{array}{rrr|r} 1 & 1 & -2 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{R_1^*=-1R_2+R_1} \left[\begin{array}{rrr|r} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

implying that there are an infinite number of solutions to the system.

(e) Here we have the augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -r-s \\ r \\ s \end{bmatrix} = r\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \ r, s \in \mathbb{R}$$

so that a basis for the solution space is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ with dimension 2.

(f) i. From part (a), we know that the roots of the characteristic equation are -2 and 1, with 1 having multiplicity 2. A basis for the solution space is thus $\{e^{-2t}, e^t, te^t\}$

ii. $y_p = Ae^{2t} + t^2(Bt^2 + Ct + D)e^t = Ae^{2t} + (Bt^4 + Ct^3 + Dt^2)e^t$

■