- This exam is worth 150 points and has 8 problems.
- Show all work and simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOUR HAVE SATISFACTORILY SCANNED AND UPLOADED YOUR EXAM TO GRADESCOPE.**
- You are taking this exam in a proctored and honor code enforced environment. No calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5"× 11" crib sheet with writing on both sides.

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0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/121923 (24 pts)] Write the word **TRUE** or **FALSE** as appropriate. Write your answers in a single column separate from any work you do. No work need be shown. No partial credit given.
   (a) The set $\mathbb{W}$ consisting of vectors of the form $\vec{\mathbf{x}} = [a \quad b \quad 0 \quad a-b]^{\mathrm{T}}$ is a subspace of $\mathbb{R}^4$.
   (b) $y = 4$ is the only equilibrium solution of $y' = t(y-4)^2$.
   (c) If the Wronskian of two arbitrary functions is identically zero on the real line, then the two functions must always be linearly dependent on $\mathbb{R}$.
   (d) If $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is consistent, then $\vec{\mathbf{b}} \in \text{Col } \mathbf{A}$.
   (e) If $\vec{\mathbf{x}}$ is an $n \times 1$ matrix and $\mathbf{A}$ is an $n \times n$ matrix, then $\vec{\mathbf{x}}^{\mathrm{T}}\mathbf{A}\vec{\mathbf{x}}$ is an $n \times 1$ matrix.
   (f) The system $\begin{cases} x' = x(3-x-2y) \\ y' = y(2-y-x) \end{cases}$ has an equilibrium solution at the origin and a v nullcline of $y = 2-x$.
   (g) Every first order linear homogeneous differential equation is separable.
   (h) If $\mathbf{A}$ is an $n \times n$ matrix with two eigenvalues equal to 0, then the columns of $\mathbf{A}$ are linearly dependent and the solution to $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$, where $\vec{\mathbf{b}}$ is an $n \times 1$ matrix, is $\vec{\mathbf{x}} = \mathbf{A}^{-1}\vec{\mathbf{b}}$.

2. [2360/121923 (18 pts)] Let $f(t) = t\,\text{step}(t) - t\,\text{step}(t-1) - 3\,\text{step}(t-2)$.
   (a) (5 pts) Write $f(t)$ as a piecewise defined function.
   (b) (5 pts) Make a well-labeled graph of $f(t)$ on the interval $[-4, 4]$.
   (c) (8 pts) Find $\mathscr{L}\{f(t)\}$.

3. [2360/121923 (20 pts)] A mass-spring system at $t = 0$ features the 1-kg mass at rest at the equilibrium position. The restoring constant is 40 N/m and the system is hooked up so that the damping force is numerically equal to 4 times the instantaneous velocity. The oscillator is subjected to a driving force of $f(t) = 40 + \delta(t-2)$. Find the displacement, $x(t)$, of the mass for all $t > 0$.

4. [2360/121923 (15 pts)] Two 100-gallon tanks are completely full. Initially, tank 1 contains 5 pounds of dissolved sugar and tank 2 has 3 pounds of dissolved sugar in it. The contents in the tanks are well stirred. The flow rate into tank 1 is always 20 gallons per minute (gpm). For $0 \le t < 4$, fresh water enters tank 1. For $t \ge 4$, the water entering tank 1 contains t pounds of sugar per gallon. For all $t \ge 0$, solution from tank 1 enters tank 2 at 25 gpm; also, solution from tank 2 enters tank 1 at 5 gpm and leaves tank 2 at 20 gpm. Set up, but **do not solve**, an initial value problem whose solution will give the amount of sugar in each tank for all time. Write your final answer using matrices and vectors.

5. [2360/121923 (12 pts) A certain object's temperature, $T(t)$, is governed by the differential equation $\dfrac{\mathrm{d}T}{\mathrm{d}t} = 2\left(te^{-2t} - T\right)$. If it's temperature when $t = 0$ is 1, find its temperature when $t = 1$.

6. [2360/121923 (18 pts)] Let $\mathbf{A} = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$.
   (a) (4 pts) Use the definition of eigenvalues/eigenvectors to show that $\lambda = i$ and $\vec{\mathbf{v}} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$ are an eigenvalue/eigenvector pair of matrix $\mathbf{A}$. No credit for using determinants.
   (b) (14 pts) Solve the initial value problem $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}},\ \vec{\mathbf{x}}(0) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, writing your answer as a single vector.

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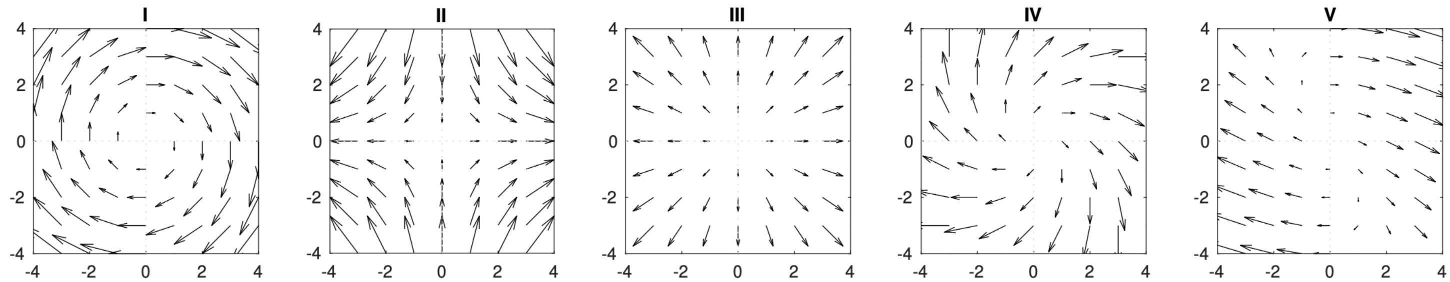
7. [2360/121923 (10 pts)] Consider the system of differential equations $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$. Match the phase portrait to the appropriate system for the given matrices. Write your answers in a single column separate from any work you do. No work need be shown. No partial credit given.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ (b) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (c) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (d) $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ (e) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$



8. [2360/121923 (33 pts)] Consider the matrix $\mathbf{A} = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix}$.

(a) (5 pts) Show that $|\mathbf{A}| = c^3 - 3c + 2$ by using the cofactor expansion method, expanding along the first row.

(b) (2 pts) Verify that $c^3 - 3c + 2 = (c+2)(c^2 - 2c + 1)$.

(c) (4 pts) Using the result in part (b), find the roots of $c^3 - 3c + 2 = 0$ and state the multiplicity of each.

(d) (12 pts) Using the information gathered in parts (a), (b), and (c), determine the number of solutions to the system $\mathbf{A}\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ if

i. $c = 1$ ii. $c \neq 1, -2$ iii. $c = -2$

(e) (5 pts) Find a basis for the solution space of $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ when $c = 1$. What is the dimension of the solution space for this case?

(f) (5 pts) Suppose the characteristic equation of a linear, homogeneous, constant coefficient differential equation is $r^3 - 3r + 2 = 0$. Use the information from parts (a), (b) and (c) to answer the following questions.

i. (3 pts) Find a basis for the solution space of the differential equation.

ii. (2 pts) Find the form of the particular solution you would use in the Method of Undetermined Coefficients if the differential equation was forced by the nonhomogeneous term $f(t) = e^{2t} + t^2 e^t$. **Do not** solve for the coefficients.

Short table of Laplace Transforms: $\mathscr{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a, b, c are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\{\cosh bt\} = \frac{s}{s^2-b^2} \qquad \mathscr{L}\{\sinh bt\} = \frac{b}{s^2-b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\{e^{at} f(t)\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{t f'(t)\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\{f(t-c)\,\text{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\text{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$
