1. [2360/112923 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. Write your answers in a single column separate from any work you do. No work need be shown. No partial credit given.

   (a) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2+4s+8}\right\} = \dfrac{1}{2}e^{2t}\sin 2t$

   (b) $\ddot{x} + x^3 - t = 0$ describes a conservative system.

   (c) The general solution of $y^{(4)} + 8y'' + 16y = 0$ is $y(t) = (c_1 + c_2 t)\cos 2t + (c_3 + c_4 t)\sin 2t$.

   (d) $\displaystyle\lim_{b\to\infty}\int_0^b t^5 e^{(4-s)t}\,\mathrm{d}t$ exists if $s > 4$.

   (e) $\{t, t\ln t\}$ is a basis for the solution space of $t^2y'' - 2ty' + y = 0,\ t > 0$.

   SOLUTION:

   (a) **FALSE** $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2+4s+8}\right\} = \dfrac{1}{2}\mathscr{L}^{-1}\left\{\dfrac{2}{(s+2)^2+4}\right\} = \dfrac{1}{2}e^{-2t}\sin 2t$

   (b) **FALSE** The equation is not autonomous.

   (c) **TRUE** The characteristic equation is $r^4 + 8r^2 + 16 = \left(r^2\right)^2 + 8r^2 + 16 = \left(r^2+4\right)^2 = 0 \implies r = \pm 2i$ each with multiplicity 2. Thus a basis for the solution space is $\{\cos 2t, t\cos 2t, \sin 2t, t\sin 2t\}$ and the general solution will have the form $(c_1 + c_2 t)\cos 2t + (c_3 + c_4 t)\sin 2t$

   (d) **TRUE** $\displaystyle\lim_{b\to\infty}\int_0^b t^5 e^{(4-s)t}\,\mathrm{d}t = \int_0^\infty t^5 e^{4t}e^{-st}\,\mathrm{d}t = \mathscr{L}\left\{t^5e^{4t}\right\} = \frac{5!}{(s-4)^6}$

   (e) **FALSE** The functions are linearly independent but are not solutions of the differential equation.

$$W[t, t\ln t] = \begin{vmatrix} t & t\ln t \\ 1 & 1+\ln t \end{vmatrix} = t \not\equiv 0 \implies \text{linear independence}$$

$$t^2(t)'' - 2t(t)' + t = 0 - 2t + t = -t \neq 0$$

$$t^2(t\ln t)'' - 2t(t\ln t)' + t\ln t = t^2t^{-1} - 2t(1+\ln t) + t\ln t = -t - t\ln t \neq 0$$

2. [2360/112923 (21 pts)] Consider a mass/spring system oriented horizontally on your desk. The mass is 2 kg, the apparatus is set up such that the damping force is equal to 3 times the instantaneous velocity, and the restoring constant is 4 N/m.

   (a) (2 pts) If the system is unforced, how many times will the mass pass through the equilibrium position if the initial displacement and/or velocity is nonzero?

   (b) (2 pts) Is the system overdamped, underdamped, or critically damped?

   (c) (3 pts) If the system is driven with a forcing function $f(t) = 10\sin\sqrt{2}t$, will the amplitude of the oscillations grow without bound? Justify your answer.

   (d) (14 pts) Now suppose the system is unforced and the spring has been replaced with one that has a restoring constant of 1 N/m. Let $x(t)$ be the displacement (in meters) from the equilibrium position and suppose $x(0) = 1$ and $\dot{x}(0) = -2$.

      i. (2 pts) Where is the mass at $t = 0$?

      ii. (2 pts) Is the mass moving at $t = 0$? If yes, in what direction?

      iii. (10 pts) Determine the time(s), if any, that the mass reaches it equilibrium position.

   SOLUTION:

   (a) The mass will pass through the equilibrium position infinitely many times. This is so because

$$\Delta = b^2 - 4mk = 3^2 - 4(2)(4) = -23 < 0,$$

   so the solution will be oscillatory (comprised of sines and/or cosines).

   (b) Underdamped, again since $\Delta < 0$.

   (c) No. Even though the frequency of the forcing function matches the circular frequency of the oscillator, the system is damped and damped systems cannot exhibit pure resonance; their solutions are bounded.

(d) i. 1 meter to the right of the equilibrium position.

ii. Yes, moving to the left at 2 m/s.

iii. The differential equation governing the motion is $2\ddot{x} + 3\dot{x} + x = 0$.

$$2r^2 + 3r + 1 = (2r+1)(r+1) = 0 \implies r = -\frac{1}{2}, -1$$

$$x(t) = c_1 e^{-t/2} + c_2 e^{-t}$$

$$\dot{x}(t) = -\frac{1}{2}c_1 e^{-t/2} - c_2 e^{-t}$$

We then apply the initial conditions, resulting in the linear system

$$x(0) = c_1 + c_2 = 1$$

$$\dot{x}(0) = -\frac{1}{2}c_1 - c_2 = -2$$

the solution of which is $c_1 = -2$ and $c_2 = 3$. The displacement is then $x(t) = -2e^{-t/2} + 3e^{-t}$. To see if/when the mass passes through the equilibrium position, we need to see if $x(t) = 0$ has any solutions.

$$e^t\left(-2e^{-t/2} + 3e^{-t} = 0\right)$$

$$-2e^{t/2} = -3$$

$$e^{t/2} = \frac{3}{2}$$

$$\frac{t}{2} = \ln\frac{3}{2}$$

$$t = 2\ln\frac{3}{2}$$

The mass indeed passes through its equilibrium position at $t = 2\ln\frac{3}{2}$ seconds.

■

3. [2360/112923 (32 pts)] Consider the linear operator $L(\vec{\mathbf{y}}) = 3y'' + 9y'$.

(a) (7 pts) Solve $L(\vec{\mathbf{y}}_h) = 0$.

(b) (10 pts) Use the Method of Undetermined Coefficients to find a particular solution of $L(\vec{\mathbf{y}}) = 54t$.

(c) (10 pts) Use Variation of Parameters to find a particular solution of $L(\vec{\mathbf{y}}) = 90e^{2t}$.

(d) (5 pts) Find the general solution of $L(\vec{\mathbf{y}}) = 54t + 90e^{2t}$.

SOLUTION:

(a) The characteristic equation is $3r^2 + 9r = 3r(r+3) = 0$ which has roots of $0, -3$ giving $y_h(t) = c_1 + c_2 e^{-3t}$.

(b) The initial guess for y_{p_1} is $At + B$, but since part of that is a solution to the homogeneous problem, we must modify it by multiplication by t. Thus $y_{p_1} = At^2 + Bt$. Substituting this into the differential equation we have

$$3y_{p_1}'' + 9y_{p_1}' = 3(2A) + 9(2At + B) = 54t$$

$$6A + 9B + 18At = 54t$$

$$\left.\begin{aligned} 18A &= 54 \\ 6A + 9B &= 0 \end{aligned}\right\} \implies A = 3 \text{ and } B = -2$$

$$y_{p_1} = 3t^2 - 2t$$

(c) We begin by getting the equation into standard form (coefficient of $y'' = 1$), $y'' + 3y' = 30e^{2t}$, giving $f = 30e^{2t}$ and note that

$y_{p_2} = v_1 y_1 + v_2 y_2$ with $y_1 = 1$ and $y_2 = e^{-3t}$ from part (a).

$$W[1, e^{-3t}](t) = \begin{vmatrix} 1 & e^{-3t} \\ 0 & -3e^{-3t} \end{vmatrix} = -3e^{-3t}$$

$$v_1' = \frac{-y_2 f}{W(t)} = \frac{-e^{-3t}\left(30e^{2t}\right)}{-3e^{-3t}} = 10e^{2t} \implies v_1 = \int 10e^{2t}\,\mathrm{d}t = 5e^{2t}$$

$$v_2' = \frac{y_1 f}{W(t)} = \frac{1\left(30e^{2t}\right)}{-3e^{-3t}} = -10e^{5t} \implies v_2 = \int -10e^{5t}\,\mathrm{d}t = -2e^{5t}$$

$$y_{p_2} = 5e^{2t}(1) - 2e^{5t}e^{-3t} = 3e^{2t}$$

(d) Use the Nonhomogeneous Principle as well as the Superposition Principle for Nonhomogeneous Linear DEs to write

$$y(t) = y_h(t) + y_p(t) = y_h(t) + y_{p_1}(t) + y_{p_2}(t) = c_1 + c_2 e^{-3t} + 3t^2 - 2t + 3e^{2t}$$

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4. [2360/112923 (14 pts) Solve the initial value problem $x' + 2x = 4,\ x(0) = 5$ using Laplace transforms and identify the transient and steady state solutions, if any exist.

SOLUTION:

$$\mathscr{L}\left\{x' + 2x = 4\right\}$$

$$sX(s) - x(0) + 2X(s) = \frac{4}{s}$$

$$X(s) = \frac{4}{s(s+2)} + \frac{5}{s+2}$$

Performing a partial fraction decomposition on the first term gives

$$\frac{4}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}$$

$$4 = A(s+2) + Bs$$

$$s = -2 : 4 = B(-2) \implies B = -2$$

$$s = 0 : 4 = A(2) \implies A = 2$$

$$\frac{4}{s(s+2)} = \frac{2}{s} - \frac{2}{s+2}$$

Then

$$X(s) = \frac{2}{s} + \frac{3}{s+2}$$

$$x(t) = \mathscr{L}^{-1}\left\{\frac{2}{s} + \frac{3}{s+2}\right\} = 2 + 3e^{-2t}$$

The transient solution is $3e^{-2t}$ and the steady state solution is 2. ■

5. [2360/112923 (23 pts)] The characteristic equation obtained from a certain homogeneous linear differential equation, $L(\vec{\mathbf{y}}) = 0$, is $\left(r^2 - 1\right)(r + 4) = 0$.

(a) (5 pts) Find a basis for the solution space of the differential equation.

(b) (5 pts) Write the original homogeneous differential equation from which the characteristic equation was derived, that is, find L.

(c) (5 pts) Convert the equation in part (b) into a system of three first order equations, writing your answer in the form $\vec{\mathbf{u}}' = \mathbf{A}\vec{\mathbf{u}}$.

(d) (8 pts) For the following forcing functions, write down the form of the particular solution that you would use to solve $L(\vec{\mathbf{y}}) = f(t)$ using the method of undetermined coefficients. **DO NOT** solve for the constants and write "NA" if that method is not applicable

i. $f(t) = \sin 3t + \sin 2t$ ii. $f(t) = te^t$ iii. $f(t) = \ln t$ iv. $f(t) = 3t^{-2}$

SOLUTION:

(a) The characteristic equation can be further factored as $(r-1)(r+1)(r+4)=0$, the roots of which are $r=-4,-1,1$. This gives a basis for the solution space as $\left\{e^{-4t}, e^{-t}, e^{t}\right\}$.

(b) Multiplying out the characteristic equation yields $r^3+4r^2-r-4=0$ so the differential equation is $y'''+4y''-y'-4y=0$.

(c)

$$u_1 = y, \quad u_2 = y', \quad u_3 = y''$$

$$u_1' = u_2, \quad u_2' = u_3 \quad u_3' = 4u_1 + u_2 - 4u_3$$

$$\begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

(d) i. $y_p = A\sin 3t + B\cos 3t + C\sin 2t + D\cos 2t$

ii. $y_p = \left(At^2 + Bt\right)e^t$

iii. N/A

iv. N/A

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