- This exam is worth 100 points and has 5 problems.
- **Show all work and simplify your answers!** Answers with no justification will receive no points unless otherwise noted.
- **Please begin each problem on a new page.**
- **DO NOT** leave the exam until you have satisfactorily scanned and uploaded your exam to Gradescope.
- You are taking this exam in a proctored and honor code enforced environment. **NO** calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5"× 11" crib sheet with writing on one side.

0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/112923 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. Write your answers in a single column separate from any work you do. No work need be shown. No partial credit given.
   (a) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2+4s+8}\right\} = \dfrac{1}{2}e^{2t}\sin 2t$
   (b) $\ddot{x} + x^3 - t = 0$ describes a conservative system.
   (c) The general solution of $y^{(4)} + 8y'' + 16y = 0$ is $y(t) = (c_1 + c_2 t)\cos 2t + (c_3 + c_4 t)\sin 2t$.
   (d) $\displaystyle\lim_{b\to\infty}\int_0^b t^5 e^{(4-s)t}\,\mathrm{d}t$ exists if $s > 4$.
   (e) $\{t, t\ln t\}$ is a basis for the solution space of $t^2y'' - 2ty' + y = 0,\ t > 0$.

2. [2360/112923 (21 pts)] Consider a mass/spring system oriented horizontally on your desk. The mass is 2 kg, the apparatus is set up such that the damping force is equal to 3 times the instantaneous velocity, and the restoring constant is 4 N/m.
   (a) (2 pts) If the system is unforced, how many times will the mass pass through the equilibrium position if the initial displacement and/or velocity is nonzero?
   (b) (2 pts) Is the system overdamped, underdamped, or critically damped?
   (c) (3 pts) If the system is driven with a forcing function $f(t) = 10\sin\sqrt{2}t$, will the amplitude of the oscillations grow without bound? Justify your answer.
   (d) (14 pts) Now suppose the system is unforced and the spring has been replaced with one that has a restoring constant of 1 N/m. Let $x(t)$ be the displacement (in meters) from the equilibrium position and suppose $x(0) = 1$ and $\dot{x}(0) = -2$.
      i. (2 pts) Where is the mass at $t = 0$?
      ii. (2 pts) Is the mass moving at $t = 0$? If yes, in what direction?
      iii. (10 pts) Determine the time(s), if any, that the mass reaches it equilibrium position.

3. [2360/112923 (32 pts)] Consider the linear operator $L(\vec{\mathbf{y}}) = 3y'' + 9y'$.
   (a) (7 pts) Solve $L(\vec{\mathbf{y}}_h) = 0$.
   (b) (10 pts) Use the Method of Undetermined Coefficients to find a particular solution of $L(\vec{\mathbf{y}}) = 54t$.
   (c) (10 pts) Use Variation of Parameters to find a particular solution of $L(\vec{\mathbf{y}}) = 90e^{2t}$.
   (d) (5 pts) Find the general solution of $L(\vec{\mathbf{y}}) = 54t + 90e^{2t}$.

4. [2360/112923 (14 pts) Solve the initial value problem $x' + 2x = 4,\ x(0) = 5$ using Laplace transforms and identify the transient and steady state solutions, if any exist.

MORE PROBLEMS AND LAPLACE TRANSFORM TABLE ON REVERSE

5. [2360/112923 (23 pts)] The characteristic equation obtained from a certain homogeneous linear differential equation, $L(\vec{\mathbf{y}}) = 0$, is $\left(r^2 - 1\right)(r+4) = 0$.

(a) (5 pts) Find a basis for the solution space of the differential equation.

(b) (5 pts) Write the original homogeneous differential equation from which the characteristic equation was derived, that is, find L.

(c) (5 pts) Convert the equation in part (b) into a system of three first order equations, writing your answer in the form $\vec{\mathbf{u}}' = \mathbf{A}\vec{\mathbf{u}}$.

(d) (8 pts) For the following forcing functions, write down the form of the particular solution that you would use to solve $L(\vec{\mathbf{y}}) = f(t)$ using the method of undetermined coefficients. **DO NOT** solve for the constants and write "NA" if that method is not applicable

i. $f(t) = \sin 3t + \sin 2t$ ii. $f(t) = te^t$ iii. $f(t) = \ln t$ iv. $f(t) = 3t^{-2}$

Short table of Laplace Transforms: $\mathscr{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a, b, c are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\{\cosh bt\} = \frac{s}{s^2-b^2} \qquad \mathscr{L}\{\sinh bt\} = \frac{b}{s^2-b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{tf'(t)\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\{f(t-c)\,\text{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\text{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$