1. [2360/102523 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

For all parts of the problem, assume $\mathbf{A}$ is a 2×5 matrix and $\mathbf{C}$ is a singular 5×5 matrix.

(a) $\left|\left(\mathbf{A}\mathbf{A}^{\mathrm{T}}\right)^2\right| \neq \left|\mathbf{A}\mathbf{A}^{\mathrm{T}}\right|^2$

(b) 0 is an eigenvalue of $\mathbf{C}$.

(c) $\mathbf{C}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ is inconsistent.

(d) $\mathbf{A} + \left(\mathbf{C}\mathbf{A}^{\mathrm{T}}\right)^{\mathrm{T}}$ is not defined.

(e) $(\text{Tr } \mathbf{C})^2 - 4|\mathbf{C}| = (\text{Tr } \mathbf{C})^2$

SOLUTION:

(a) **FALSE** $\left|\left(\mathbf{A}\mathbf{A}^{\mathrm{T}}\right)^2\right| = \left|\left(\mathbf{A}\mathbf{A}^{\mathrm{T}}\right)\left(\mathbf{A}\mathbf{A}^{\mathrm{T}}\right)\right| = \left|\mathbf{A}\mathbf{A}^{\mathrm{T}}\right|\left|\mathbf{A}\mathbf{A}^{\mathrm{T}}\right| = \left|\mathbf{A}\mathbf{A}^{\mathrm{T}}\right|^2$ since $\mathbf{A}\mathbf{A}^{\mathrm{T}}$ is square (2×2)

(b) **TRUE** Since $\mathbf{C}$ is singular, at least one of its eigenvalues must be zero.

(c) **FALSE** Homogeneous systems are always consistent since they have at least the trivial solution $\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

(d) **FALSE** $\left(\mathbf{C}\mathbf{A}^{\mathrm{T}}\right)^{\mathrm{T}} = \left(\mathbf{A}^{\mathrm{T}}\right)^{\mathrm{T}}\mathbf{C}^{\mathrm{T}} = \mathbf{A}\mathbf{C}^{\mathrm{T}}$ which is 2×5, the same size/order as $\mathbf{A}$ and thus the sum can be computed.

(e) **TRUE** Since $\mathbf{C}$ is singular, $|\mathbf{C}| = 0$

2. [2360/102523 (25 pts)] Let $\mathbf{B} = \begin{bmatrix} 2 & 0 & 0 \\ 4 & -1 & 0 \\ 6 & 3 & 1 \end{bmatrix}$.

(a) (10 pts) Use Gauss-Jordan elimination to find $\mathbf{B}^{-1}$.

(b) (5 pts) State the definition for $n \times n$ matrix $\mathbf{H}$ to be the inverse of $n \times n$ matrix $\mathbf{G}$ (there are two requirements). Verify that your answer to part (a) is correct by showing that one of the requirements in the definition holds.

(c) (6 pts) Use your answer to part (a) to solve the system $\mathbf{B}\vec{\mathbf{y}} = \vec{\mathbf{c}}$ where $\vec{\mathbf{c}} = \begin{bmatrix} 4 & 10 & 6 \end{bmatrix}^{\mathrm{T}}$.

(d) (4 pts) Is Col $\mathbf{B} = \mathbb{R}^3$? Explain briefly in words.

SOLUTION:

(a)

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 4 & -1 & 0 & 0 & 1 & 0 \\ 6 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3^* = -3R_1 + R_3]{R_2^* = -2R_1 + R_2} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & 3 & 1 & -3 & 0 & 1 \end{array}\right] \xrightarrow[\substack{R_2^* = -R_2 \\ R_1^* = \frac{1}{2}R_1}]{R_3^* = 3R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -9 & 3 & 1 \end{array}\right]$$

$$\Longrightarrow \mathbf{B}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 2 & -1 & 0 \\ -9 & 3 & 1 \end{bmatrix}$$

(b) Definition: $\mathbf{H}$ is the inverse of $\mathbf{G}$, both $n \times n$ matrices, if and only if $\mathbf{G}\mathbf{H} = \mathbf{H}\mathbf{G} = \mathbf{I}$. Either one of the following will verify the answer to part (a).

$$\mathbf{B}^{-1}\mathbf{B} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 2 & -1 & 0 \\ -9 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 4 & -1 & 0 \\ 6 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

$$\mathbf{B}\mathbf{B}^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 4 & -1 & 0 \\ 6 & 3 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 2 & -1 & 0 \\ -9 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

(c)

$$\vec{\mathbf{y}} = \mathbf{B}^{-1}\vec{\mathbf{c}} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 2 & -1 & 0 \\ -9 & 3 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 10 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

(d) Yes. Since $\mathbf{B}$ is invertible, the system $\mathbf{B}\vec{\mathbf{y}} = \vec{\mathbf{c}}$ is consistent for all $\vec{\mathbf{c}}$, implying the columns of $\mathbf{B}$ span all of $\mathbb{R}^3$, that is, $\text{Col}\,\mathbf{B} = \mathbb{R}^3$.

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3. [2360/102523 (16 pts)] Consider the system

$$x_1 + 6x_2 - 2x_3 + 3x_4 = 5$$
$$3x_1 + 17x_2 - 6x_3 + 9x_4 = 13$$

(a) (8 pts) Find the solution to the system, writing your answer using the Nonhomogeneous Principle.

(b) (8 pts) Find a basis for the solution space of the associated homogeneous system. What is its dimension?

SOLUTION:

(a)

$$\left[\begin{array}{cccc|c} 1 & 6 & -2 & 3 & 5 \\ 3 & 17 & -6 & 9 & 13 \end{array}\right] \xrightarrow{R_2^*=-3R_1+R_2} \left[\begin{array}{cccc|c} 1 & 6 & -2 & 3 & 5 \\ 0 & -1 & 0 & 0 & -2 \end{array}\right] \xrightarrow[R_2^*=-R_2]{R_1^*=6R_2+R_1} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & -7 \\ 0 & 1 & 0 & 0 & 2 \end{array}\right]$$

$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -7+2s-3t \\ 2 \\ s \\ t \end{bmatrix}$$

$$\vec{\mathbf{x}} = \vec{\mathbf{x}}_h + \vec{\mathbf{x}}_p = \begin{bmatrix} 2s-3t \\ 0 \\ s \\ t \end{bmatrix} + \begin{bmatrix} -7 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad s,t \in \mathbb{R}$$

(b)

$$\vec{\mathbf{x}}_h = \begin{bmatrix} 2s-3t \\ 0 \\ s \\ t \end{bmatrix} = \begin{bmatrix} 2s \\ 0 \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} -3t \\ 0 \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

A basis for the solution space of the homogeneous system is $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ which has dimension 2.

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4. [2360/102523 (15 pts)] Let $\mathbf{F} = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

(a) (5 pts) Using the definition of what it means to be an eigenvalue/eigenvector pair (do not use a determinant), find the eigenvalue associated the eigenvector $\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

(b) (10 pts) The other eigenvalue is -2. What is its algebraic multiplicity? Find its geometric multiplicity as well as a basis for and the dimension of its eigenspace.

SOLUTION:

(a) The definition is $\mathbf{F}\vec{\mathbf{v}}_1 = \lambda\vec{\mathbf{v}}_1$

$$\begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \implies \lambda = 4$$

(b) Since there is only one other eigenvalue and we have a 3×3 matrix, its algebraic multiplicity is 2. We now seek nontrivial solutions of $(\mathbf{F} + 2\mathbf{I})\,\vec{\mathbf{v}} = \vec{\mathbf{0}}$

$$\left[\begin{array}{ccc|c} 3 & 3 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \begin{array}{l} v_1 = -v_2 = -s \\ v_2 = s \\ v_3 = t \end{array}$$

The geometric multiplicity is 2 and the dimension of the eigenspace is 2. A basis is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

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5. [2360/102523 (18 pts) Determine which of the following subsets of $\mathbb{M}_{33}$ are subspaces. Justify your answers.

(a) (6 pts) The subset, $\mathbb{W}$, of matrices of the form $\begin{bmatrix} a & 0 & a \\ 0 & a+2 & 0 \\ a & 0 & a \end{bmatrix}$ where $a \in \mathbb{R}$.

(b) (6 pts) The subset, $\mathbb{W}$, of 3×3 skew symmetric matrices $\left(\mathbf{A}^{\mathrm{T}} = -\mathbf{A}\right)$.

(c) (6 pts) The subset, $\mathbb{W}$, of 3×3 upper triangular matrices with rational number entries.

SOLUTION:

(a) Not a subspace. The zero vector, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, is not in the subset.

(b) Subspace. Subset is closed with respect to linear combinations. Assume $\mathbf{A}, \mathbf{B} \in \mathbb{W}$ and let $\alpha, \beta \in \mathbb{R}$. Then

$$(\alpha\mathbf{A} + \beta\mathbf{B})^{\mathrm{T}} = \alpha\mathbf{A}^{\mathrm{T}} + \beta\mathbf{B}^{\mathrm{T}} = \alpha(-\mathbf{A}) + \beta(-\mathbf{B}) = -(\alpha\mathbf{A} + \beta\mathbf{B}) \in \mathbb{W}$$

Alternatively, let $\vec{\mathbf{u}} = \begin{bmatrix} 0 & u_1 & u_2 \\ -u_1 & 0 & u_3 \\ -u_2 & -u_3 & 0 \end{bmatrix}$ and $\vec{\mathbf{v}} = \begin{bmatrix} 0 & v_1 & v_2 \\ -v_1 & 0 & u_3 \\ -v_2 & -v_3 & 0 \end{bmatrix}$ be in $\mathbb{W}$ and $a, b \in \mathbb{R}$. Then

$$a\vec{\mathbf{u}} + b\vec{\mathbf{v}} = a \begin{bmatrix} 0 & u_1 & u_2 \\ -u_1 & 0 & u_3 \\ -u_2 & -u_3 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & v_1 & v_2 \\ -v_1 & 0 & v_3 \\ -v_2 & -v_3 & v_3 \end{bmatrix} = \begin{bmatrix} 0 & au_1 + bv_1 & au_2 + bv_2 \\ -(au_1 + bv_1) & 0 & au_3 + bv_3 \\ -(au_2 + bv_2) & -(au_3 + bv_3) & 0 \end{bmatrix} \in \mathbb{W}$$

(c) Not a subspace. If $c \in \mathbb{R}$ is an irrational number and $\mathbf{A}$ is in the subset, $c\mathbf{A}$ is not always in the subset since the product of a rational number and an irrational number is not necessarily rational [*e.g.* $(\pi)(1/2) = \pi/2$], implying that the subset is not closed under scalar multiplication

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6. [2360/102523 (16 pts)] The following parts are not related. Both parts require complete justification.

(a) (8 pts) Is the set $\left\{e^t, t, t^2\right\}$ linearly dependent or independent on the real line?

(b) (8 pts) Does span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \right\} = \mathbb{R}^3$?

SOLUTION:

(a) Linearly independent.

$$W[e^t, t, t^2] = \begin{vmatrix} e^t & t & t^2 \\ e^t & 1 & 2t \\ e^t & 0 & 2 \end{vmatrix} = t(-1)^{1+2} \begin{vmatrix} e^t & 2t \\ e^t & 2 \end{vmatrix} + 1(-1)^{2+2} \begin{vmatrix} e^t & t^2 \\ e^t & 2 \end{vmatrix} = -t\left(2e^t - 2te^t\right) + 2e^t - t^2 e^t = e^t\left(t^2 - 2t + 2\right) \not\equiv 0$$

(b) Yes. There are three vectors in a dimension 3 vector space. If they are linearly independent then they span the space. We need to see if the trivial solution $(c_1 = c_2 = c_3 = 0)$ is the only solution to

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or equivalently} \quad \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now,

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & -3 \end{vmatrix} = 1(-1)^{1+1}\begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} + 2(-1)^{1+2}\begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix} + 1(-1)^{1+3}\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= 1(3-1) - 2(-6-1) + (1)(2+1) = 19 \neq 0$$

indicating that the trivial solution is unique. Thus the vectors are linearly independent and span $\mathbb{R}^3$.

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