1. [2360/092723 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) If $x_1(z)$ and $x_2(z)$ solve the equation $\dfrac{\mathrm{d}x}{\mathrm{d}z} = e^z x + xz$, then $x_3(z) = x_1(z) + 3x_2(z)$ is also a solution.

   (b) The equation $y' - (y-5)^2 = -1$ has a stable equilibrium solution at $y = 5$.

   (c) The form of the particular solution when using variation of parameters (Euler-Lagrange two step method) to solve the equation $9t^2x' = 3tx - \cos t, t > 0$ is $x_p = v(t)\sqrt[3]{t}$.

   (d) Isoclines of autonomous first order differential equations are always horizontal lines.

   (e) The solution to a certain differential equation, given by $y(t) = \cos t + e^{-2t}\sin t$, has no steady state solution.

SOLUTION:

(a) **TRUE** The equation is linear and homogeneous $\left[\dfrac{\mathrm{d}x}{\mathrm{d}z} - (e^z + z)\,x = 0\right]$ so the Superposition Principle applies.

(b) **FALSE** $y = 4$ (stable) and $y = 6$ (unstable)

(c) **TRUE** The homogeneous equation gives

$$\frac{\mathrm{d}x}{x} = \frac{1}{3t}$$

$$\ln|x| = \frac{1}{3}\ln|t| + k = \ln t^{1/3} + k \quad (t > 0)$$

$$|x| = e^k t^{1/3}$$

$$x = Ct^{1/3}$$

From this then $y_p = v(t)\sqrt[3]{t}$.

(d) **TRUE** Autonomous equations can be written in the form $y' = f(y)$. Isoclines are given by $f(y) =$ constant, whose solutions, if they exist, are independent of t and thus are horizontal lines.

(e) **FALSE** The steady state solution is $\cos t$ since it is bounded (periodic) as t goes to infinity.

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2. [2360/092723 (25 pts)] Consider the initial value problem $y' = \left(t^2 - 4\right)(y-1)^{4/5}$, $y(1) = 2$.

   (a) (3 pts) What is the order of the equation? Is it autonomous? Is it linear, separable, both or neither?

   (b) (3 pts) Find all equilibrium solutions.

   (c) (3 pts) For what values of t and y will the solution be horizontal?

   (d) (8 pts) Estimate the value of $y(1.01)$ using one step of Euler's Method.

   (e) (8 pts) Does Picard's theorem guarantee a unique solution to the initial value problem? Why or why not?

SOLUTION:

(a) first order, nonautonomous, separable

(b) $y = 1$

(c) The solution will be horizontal when $y' = 0$, which occurs when $y = 1$ and/or $t = \pm 2$

(d) We have $h = 0.01$.

$$y(1.01) = y(1) + 0.01[(1^2 - 4)(2-1)^{4/5} = 2 + 0.01(-3) = 1.97$$

(e) Yes. $f(t,y) = \left(t^2 - 4\right)(y-1)^{4/5}$ is continuous for all t and y and thus continuous in a rectangle surrounding the initial value $(1,2)$. $f_y(t,y) = \dfrac{4}{5}\left(t^2 - 4\right)(y-1)^{-1/5}$. This, too, is continuous in a rectangle surrounding $(1,2)$ (as long as $y > 1$). Thus, a unique solution to the initial value problem is guaranteed to exist on an interval surrounding $t = 1$ by Picard's theorem.

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3. [2360/092723 (10 pts)] Find the implicit solution to $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{e^{x-y}}{y+3}$ passing through the point $(1, 1)$.

SOLUTION:

The equation is separable.

$$y' = \frac{e^x e^{-y}}{y+3}$$

$$\int e^y (y+3)\,\mathrm{d}y = \int e^x\,\mathrm{d}x$$

$$e^y(y-1) + 3e^y = e^x + C$$

$$e^y(y+2) = e^x + C$$

$$e^1(3) = e^1 + C$$

$$2e = C$$

$$e^y(y+2) = e^x + 2e$$

■

4. [2360/092723 (23 pts)] A certain species of caterpillar lives in a meadow. The caterpillars are a favorite snack for several birds that also inhabit the meadow. Let $y(t)$ represent the population of the caterpillars, given in thousands of caterpillars, that is, $y = 1$ means there are 1000 caterpillars present. The caterpillar population grows exponentially, but the birds eat the caterpillars at a rate of ty^2, where t is the time in weeks. You want to determine if the caterpillar population can sustain itself over time. At the beginning of your observation period $(t = 0)$ you note that there are 1000 caterpillars in the meadow. The differential equation that describes the caterpillar population is $y' = y - ty^2$, a Bernoulli equation (a nonlinear equation which was studied in the homework).

(a) (5 pts) Show that the change of variable $w = y^{-1}$ transforms the Bernoulli equation into the equation $w' = -w + t$.

(b) (8 pts) Use the integrating factor method to find the general solution of the equation from part (a).

(c) (6 pts) Using your answer to part (b), find an expression for the number of caterpillars as a function of time.

(d) (4 pts) Will the caterpillar population sustain itself over a long time? Justify your answer.

SOLUTION:

(a) With $w = y^{-1}, y = w^{-1}$ and $w' = -y^{-2}y' \implies y' = -w'w^{-2}$. Using this in the original Bernoulli equation gives

$$y' = y - ty^2$$

$$-w'w^{-2} = w^{-1} - tw^{-2}$$

$$w' = -w + t$$

(b) The equation from part (a) is $w' + w = t$. With $p(t) = 1$, the integrating factor is $\mu(t) = e^t$. Thus

$$\int \left(we^t\right)'\,\mathrm{d}t = \int te^t\,\mathrm{d}t \qquad (\mathrm{d}v = e^t\,\mathrm{d}t \;\; u = t)$$

$$we^t = (t-1)e^t + C$$

$$w = t - 1 + Ce^{-t}$$

(c) Going back to the original variable, we have

$$y^{-1} = t - 1 + Ce^{-t} \implies y(t) = \frac{1}{t - 1 + Ce^{-t}}$$

to which we apply the initial condition of $y(0) = 1$ to yield

$$y(0) = 1 = \frac{1}{0 - 1 + Ce^{-0}} \implies C = 2$$

giving the number of caterpillars as a function of time as

$$y(t) = \frac{1}{2e^{-t} + t - 1}$$

(d) No. The caterpillar population will eventually vanish, implying that it cannot sustain itself. This follows from the fact that

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{1}{2e^{-t}+t-1} = 0$$

■

5. [2360/092723 (12 pts) A 10-gallon tank starts out half full of pure water at time $t = 0$. Assume that the contents of the tank are always well stirred. The inflow rate from $t = 0$ onward is 3 gallons/second and this inflow contains a (decreasing) amount of $1/(t+1)$ ounces of salt per gallon. The tank is old and it leaks badly, with the salty water leaking from the tank at $1/2$ gallon per second. Let $x(t)$ denote the total amount of salt in the tank. Write down, but **do not solve**, the initial value problem that describes $x(t)$ for $0 \le t \le T$. Determine the value of T.

SOLUTION:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \text{flow in} - \text{flow out} = 3 - \frac{1}{2} = \frac{5}{2}$$
$$V(t) = \frac{5t}{2} + C, \quad V(0) = 5 \implies V(t) = \frac{5t}{2} + 5$$

$$\frac{\mathrm{d}x}{\mathbf{d}t} = \text{mass rate in} - \text{mass rate out} = \frac{1}{t+1}(3) - \frac{x}{5t/2+5}\left(\frac{1}{2}\right)$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{x}{5t+10} = \frac{3}{t+1}$$

Since the tank initially contains pure water, $x(0) = 0$. This initial value problem and its solution will be valid until the tank fills at $t = T$, which will occur when

$$V(T) = \frac{5T}{2} + 5 = 10 \implies T = 2$$

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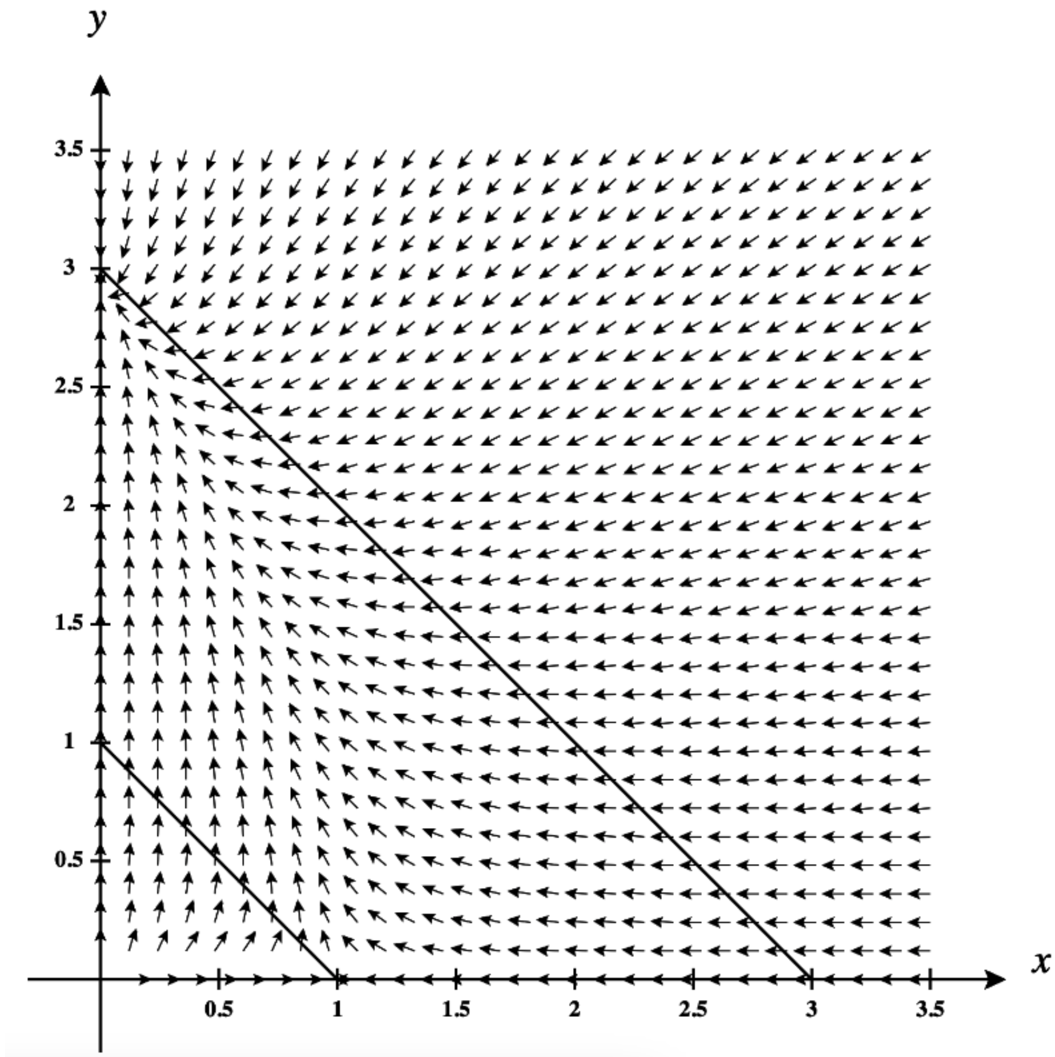
6. [2360/092723 (20 pts)] Let $x(t)$ represent the number of sheep (in thousands) in a certain pasture and $y(t)$ the number of goats (in thousands) in the same pasture. Both species compete for the same grass in the pasture. Suppose the competition model governing the two species is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(1-x-y)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = y(3-x-y)$$

The figure below should be sufficient to answer the following questions, but you are free to use the equations themselves if you want.



(a) (2 pts) Is the y-axis a v or h nullcline?

(b) (2 pts) Is the x-axis a v or h nullcline?

(c) (3 pts) What is the rate of change with respect to time of the sheep for points (states of the system) on the lower slanted line?

(d) (3 pts) Is the goat population increasing, decreasing, or remaining constant with time for points (states of the system) on the upper slanted line?

(e) (6 pts) Find the equilibrium points, if any exist, and determine their stability.

(f) (4 pts) If any goats are present initially, can the sheep survive? Explain in just a few words why or why not.

SOLUTION:

(a) The y-axis $(x = 0)$ is a v nullcline since $x' = 0$ there.

(b) The x-axis $(y = 0)$ is an h nullcline since $y' = 0$ there.

(c) The lower line, $y = 1 - x$, is a v nullcline so the rate of change of the sheep (x') is 0 at points on that line.

(d) The upper slanted line, $y = 3 - x$, is an h nullcline so the goat population is constant at points on that line $(y' = 0)$.

(e) Equilibrium points are where the h and v nullclines intersect. From the figure, these are:

- $(0, 0)$ (unstable)
- $(1, 0)$ (unstable)
- $(0, 3)$ (stable)

(f) No. All trajectories for initial states where $y(0) > 0$ (some goats present) are attracted to the equilibrium solution $(0, 3)$, indicating that the goats drive the sheep to extinction.

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