- This exam is worth 100 points and has 6 problems.
- **Show all work and simplify your answers!** Answers with no justification will receive no points unless otherwise noted.
- **Please begin each problem on a new page.**
- **DO NOT** leave the exam until you have satisfactorily scanned and uploaded your exam to Gradescope.
- You are taking this exam in a proctored and honor code enforced environment. **NO** calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5"× 11" crib sheet with writing on one side.

0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/092723 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) If $x_1(z)$ and $x_2(z)$ solve the equation $\dfrac{\mathrm{d}x}{\mathrm{d}z} = e^z x + xz$, then $x_3(z) = x_1(z) + 3x_2(z)$ is also a solution.

   (b) The equation $y' - (y-5)^2 = -1$ has a stable equilibrium solution at $y = 5$.

   (c) The form of the particular solution when using variation of parameters (Euler-Lagrange two step method) to solve the equation $9t^2x' = 3tx - \cos t, t > 0$ is $x_p = v(t)\sqrt[3]{t}$.

   (d) Isoclines of autonomous first order differential equations are always horizontal lines.

   (e) The solution to a certain differential equation, given by $y(t) = \cos t + e^{-2t}\sin t$, has no steady state solution.

2. [2360/092723 (25 pts)] Consider the initial value problem $y' = \left(t^2 - 4\right)(y-1)^{4/5},\ y(1) = 2$.

   (a) (3 pts) What is the order of the equation? Is it autonomous? Is it linear, separable, both or neither?

   (b) (3 pts) Find all equilibrium solutions.

   (c) (3 pts) For what values of t and y will the solution be horizontal?

   (d) (8 pts) Estimate the value of $y(1.01)$ using one step of Euler's Method.

   (e) (8 pts) Does Picard's theorem guarantee a unique solution to the initial value problem? Why or why not?

3. [2360/092723 (10 pts)] Find the implicit solution to $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{e^{x-y}}{y+3}$ passing through the point $(1, 1)$.

4. [2360/092723 (23 pts)] A certain species of caterpillar lives in a meadow. The caterpillars are a favorite snack for several birds that also inhabit the meadow. Let $y(t)$ represent the population of the caterpillars, given in thousands of caterpillars, that is, $y = 1$ means there are 1000 caterpillars present. The caterpillar population grows exponentially, but the birds eat the caterpillars at a rate of ty^2, where t is the time in weeks. You want to determine if the caterpillar population can sustain itself over time. At the beginning of your observation period ($t = 0$) you note that there are 1000 caterpillars in the meadow. The differential equation that describes the caterpillar population is $y' = y - ty^2$, a Bernoulli equation (a nonlinear equation which was studied in the homework).

   (a) (5 pts) Show that the change of variable $w = y^{-1}$ transforms the Bernoulli equation into the equation $w' = -w + t$.

   (b) (8 pts) Use the integrating factor method to find the general solution of the equation from part (a).

   (c) (6 pts) Using your answer to part (b), find an expression for the number of caterpillars as a function of time.

   (d) (4 pts) Will the caterpillar population sustain itself over a long time? Justify your answer.

5. [2360/092723 (12 pts) A 10-gallon tank starts out half full of pure water at time $t = 0$. Assume that the contents of the tank are always well stirred. The inflow rate from $t = 0$ onward is 3 gallons/second and this inflow contains a (decreasing) amount of $1/(t+1)$ ounces of salt per gallon. The tank is old and it leaks badly, with the salty water leaking from the tank at $1/2$ gallon per second. Let $x(t)$ denote the total amount of salt in the tank. Write down, but **do not solve**, the initial value problem that describes $x(t)$ for $0 \le t \le T$. Determine the value of T.

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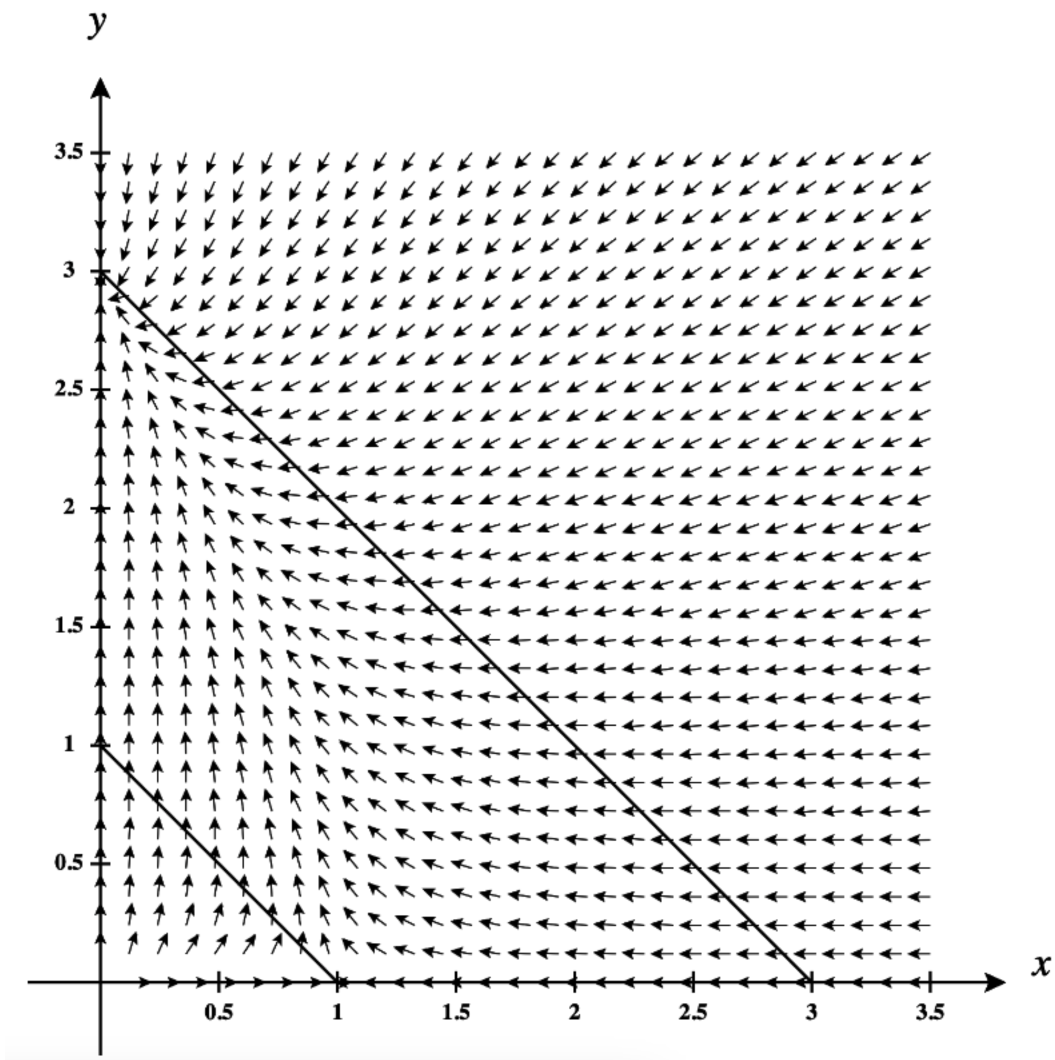
6. [2360/092723 (20 pts)] Let $x(t)$ represent the number of sheep (in thousands) in a certain pasture and $y(t)$ the number of goats (in thousands) in the same pasture. Both species compete for the same grass in the pasture. Suppose the competition model governing the two species is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(1 - x - y)$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y(3 - x - y)$$

The figure below should be sufficient to answer the following questions, but you are free to use the equations themselves if you want.



(a) (2 pts) Is the y-axis a v or h nullcline?

(b) (2 pts) Is the x-axis a v or h nullcline?

(c) (3 pts) What is the rate of change with respect to time of the sheep for points (states of the system) on the lower slanted line?

(d) (3 pts) Is the goat population increasing, decreasing, or remaining constant with time for points (states of the system) on the upper slanted line?

(e) (6 pts) Find the equilibrium points, if any exist, and determine their stability.

(f) (4 pts) If any goats are present initially, can the sheep survive? Explain in just a few words why or why not.