1. [2360/111622 (18 pts)] Consider the differential equation $ty'' + (2t-1)y' - 2y = \dfrac{48t^3}{e^{2t}},\ t > 0.$

   (a) (6 pts) Are $y_1 = e^{-2t}$ and $y_2 = 2t - 1$ linearly independent solutions of the associated homogeneous equation? Justify your answer completely.

   (b) (12 pts) Find the general solution of the equation.

SOLUTION:

(a) Yes.

$$t\left(e^{-2t}\right)'' + (2t-1)\left(e^{-2t}\right)' - 2e^{-2t} = 4te^{-2t} - 4te^{-2t} + 2e^{-2t} - 2e^{-2t} = 0$$

$$t\,(2t-1)'' + (2t-1)\,(2t-1)' - 2(2t-1) = 0 + 4t - 2 - 4t + 2 = 0$$

$$W[e^{-2t}, 2t-1] = \begin{vmatrix} e^{-2t} & 2t-1 \\ -2e^{-2t} & 2 \end{vmatrix} = 4te^{-2t} \neq 0$$

(b) The equation can be rewritten as $y'' + \dfrac{(2t-1)}{t}y' - \dfrac{2}{t}y = 48t^2e^{-2t}$. The particular solution will be in the form $y_p = v_1y_1 + v_2y_2$ with

$$v_1 = -\int \frac{(2t-1)(48t^2e^{-2t})}{4te^{-2t}}\mathrm{d}t = -12\int (2t^2 - t)\,\mathrm{d}t = -12\left(\frac{2t^3}{3} - \frac{t^2}{2}\right) = -8t^3 + 6t^2$$

$$v_2 = \int \frac{(e^{-2t})(48t^2e^{-2t})}{4te^{-2t}}\mathrm{d}t = 12\int te^{-2t}\,\mathrm{d}t = 12\left(-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t}\right) = -6te^{-2t} - 3e^{-2t}$$

$$y_p = \left(-8t^3 + 6t^2\right)e^{-2t} + \left(-6te^{-2t} - 3e^{-2t}\right)(2t-1) = e^{-2t}\left(-8t^3 - 6t^2 + 3\right)$$

The general solution is then $y(t) = c_1e^{-2t} + c_2(2t-1) + e^{-2t}\left(-8t^3 - 6t^2 + 3\right) = c_1e^{-2t} + c_2(2t-1) + e^{-2t}\left(-8t^3 - 6t^2\right)$ where the 3 in the last term was absorbed into c_1.

■

2. [2360/111622 (15 pts)] Solve the initial value problem $\dfrac{y'}{y^2} - \dfrac{3}{xy} = 1,\ y(2) = 2,\ x > 0$ by using the substitution $u(x) = \dfrac{1}{y(x)}$.

SOLUTION:

$$u' = -\frac{y'}{y^2}$$

$$-u' - \frac{3}{x}u = 1 \implies u' + \frac{3}{x}u = -1$$

integrating factor $\mu(x) = e^{\int \frac{3}{x}\mathrm{d}x} = x^3$

$$\left(x^3u\right)' = -x^3$$

$$x^3u = -\frac{x^4}{4} + c$$

$$u = -\frac{x}{4} + \frac{c}{x^3}$$

$$y = \left(\frac{c}{x^3} - \frac{x}{4}\right)^{-1}$$

apply initial condition

$$y(2) = \left(\frac{c}{8} - \frac{2}{4}\right)^{-1} = 2$$

$$\frac{c}{8} - \frac{1}{2} = \frac{1}{2} \implies c = 8$$

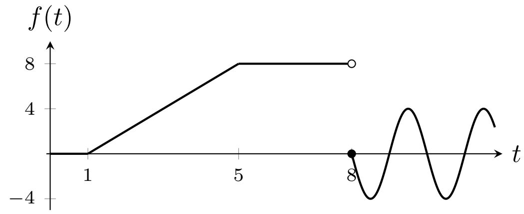
$$y(x) = \left(\frac{8}{x^3} - \frac{x}{4}\right)^{-1}$$

3. [2360/111622 (22 pts)] Your niece and nephew have had a lot of fun playing with the harmonic oscillator toy you gave them as a present at the time of the last exam. They have played around with it so much that the damping coefficient and spring/restoring constant have both increased. They have also managed to lose all but two of the stones that came with the toy but the mass of the bucket attached to the spring remains at 1 kg. As a result, the differential equation governing the toy is now $3\ddot{x} + 12\dot{x} + 9x = f(t)$.

   (a) (8 pts) Find the equation of motion if the toy is unforced and initially given an 8 m/s push to the right from a starting position 2 m to the left of the equilibrium position.

   (b) (4 pts) The kids are very interested in knowing the exact time that the bucket will return to the equilibrium position. Provide them (and your grader) an answer.

   (c) (10 pts) To increase their interest in becoming engineers, you have an addition to the toy which provides a driving force, $f(t)$, shown in the following figure. The oscillating portion is given by $-4\sin \pi t$ for $t \geq 8$. The kids (and your grader) want to see the driving force written as a single function using step functions.



SOLUTION:

(a) The characteristic equation is $3r^2 + 12r + 9 = 3(r^2 + 4r + 3) = 3(r+1)(r+3) = 0 \implies r = -1, -3$. Writing the general solution and then applying the initial conditions yields

$$x(t) = c_1 e^{-t} + c_2 e^{-3t}$$
$$x(0) = c_1 + c_2 = -2$$
$$\dot{x}(0) = -c_1 - 3c_2 = 8$$
$$\implies c_1 = 1, c_2 = -3$$

So the equation of motion is $x(t) = e^{-t} - 3e^{-3t}$.

(b) The bucket will be at the equilibrium position when $x(t) = 0$. This means

$$e^{-t} - 3e^{-3t} = 0$$
$$\left[e^{-t} = 3e^{-3t}\right] e^{3t}$$
$$e^{2t} = 3$$
$$t = \frac{1}{2}\ln 3$$

(c)
$$f(t) = (2t-2)\left[\text{step}(t-1) - \text{step}(t-5)\right] + 8\left[\text{step}(t-5) - \text{step}(t-8)\right] - 4\sin(\pi t)\text{step}(t-8)$$

4. [2360/111622 (18 pts)] Find the general solution of the following system, writing your answer in the form prescribed by the Nonhomogeneous Principle.

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 3 \\ 2x_1 + 4x_2 - 12x_3 &= 4 \\ 2x_1 + x_2 + 12x_3 &= 7 \\ 3x_1 + 3x_2 + 6x_3 &= 9 \end{aligned}$$

SOLUTION:

$$\left[\begin{array}{rrr|r} 1 & 1 & 2 & 3 \\ 2 & 4 & -12 & 4 \\ 2 & 1 & 12 & 7 \\ 3 & 3 & 6 & 9 \end{array}\right] \xrightarrow[R_4^*=-3R_1+R_4]{\substack{R_2^*=-2R_1+R_2 \\ R_3^*=-2R_1+R_3}} \left[\begin{array}{rrr|r} 1 & 1 & 2 & 3 \\ 0 & 2 & -16 & -2 \\ 0 & -1 & 8 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1^*=-1R_2+R_1]{\substack{R_2^*=\frac{1}{2}R_2 \\ R_3^*=R_2+R_3}} \left[\begin{array}{rrr|r} 1 & 0 & 10 & 4 \\ 0 & 1 & -8 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This RREF gives $x_3 = t$ as the free parameter with $x_1 = 4 - 10t$ and $x_2 = -1 + 8t$ so that, written in the form $\vec{\mathbf{x}} = \vec{\mathbf{x}}_h + \vec{\mathbf{x}}_p$, we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -10 \\ 8 \\ 1 \end{bmatrix} t}_{\vec{\mathbf{x}}_h} + \underbrace{\begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}}_{\vec{\mathbf{x}}_p}, \quad t \in \mathbb{R}$$

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5. [2360/111622 (16 pts)] Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 5 & 0 & 0 \end{bmatrix}$.

(a) (8 pts) Find the eigenvalues and eigenvectors of $\mathbf{A}$.

(b) (8 pts) Solve $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \begin{bmatrix} 2 & -1 & -5 \end{bmatrix}^{\mathrm{T}}$. Write your answer as a single vector.

SOLUTION:

(a) Since the matrix is lower triangular, the eigenvalues lie on the diagonal and are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 0$. For each $i = 1, 2, 3$ we need to find nontrivial solutions to $(\mathbf{A} - \lambda_i \mathbf{I})\vec{\mathbf{v}}_i = \mathbf{0}$.

$$\lambda_1 = 1: \quad \left[\begin{array}{rrr|r} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 5 & 0 & -1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{rrr|r} 1 & 0 & -\frac{1}{5} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$$

$$\lambda_2 = 2: \quad \left[\begin{array}{rrr|r} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 5 & 0 & -2 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \vec{\mathbf{v}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 0: \quad \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \vec{\mathbf{v}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(b) Using the eigenvectors and eigenvalues from part (a), the general solution is

$$\vec{\mathbf{x}} = c_1 e^t \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

to which we apply the initial condition to get

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix} \implies \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -15 \end{bmatrix}$$

so that

$$\vec{\mathbf{x}} = 2e^t \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} - e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 15 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^t \\ -e^{2t} \\ 5(2e^t - 3) \end{bmatrix}$$

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6. [2360/111622 (20 pts)] Solve $y'' + 4y' + 6y = \delta(t-2)$ $y(0) = 2$, $y'(0) = -1$.

SOLUTION:

Taking the Laplace Transform of both sides gives

$$s^2 Y(s) - sy(0) - y'(0) + 4[sY(s) - y(0)] + 6Y(s) = e^{-2s}$$

$$\left(s^2 + 4s + 6\right) Y(s) = e^{-2s} + sy(0) + y'(0) + 4y(0) = e^{-2s} + 2s + 7 = e^{-2s} + 2(s+2) + 3$$

$$Y(s) = \frac{1}{\sqrt{2}} \frac{\sqrt{2} e^{-2s}}{(s+2)^2 + 2} + 2\frac{(s+2)}{(s+2)^2+2} + \frac{3}{\sqrt{2}} \frac{\sqrt{2}}{(s+2)^2+2}$$

$$y(t) = \mathscr{L}^{-1} Y(s) = \frac{1}{\sqrt{2}} e^{-2(t-2)} \sin\left[\sqrt{2}(t-2)\right] \operatorname{step}(t-2) + 2e^{-2t}\cos\sqrt{2}t + \frac{3}{\sqrt{2}} e^{-2t} \sin\sqrt{2}t$$

7. [2360/111622 (21 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

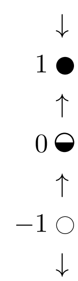
(a) The set $\mathbb{W}$ of all solutions to $2x_1 - 5x_2 + 7x_3 - 1 = 0$ forms a 2-dimensional subspace of $\mathbb{R}^3$.

(b) The h- and v-nullclines of the system $\vec{\mathbf{x}}' = \begin{bmatrix} 2 & -2 \\ 15 & -7 \end{bmatrix} \vec{\mathbf{x}} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are lines through the origin.

(c) All $n \times n$ matrices that have two identical columns are singular.

(d) $L(\vec{\mathbf{y}})$ represents a constant coefficient differential operator. The characteristic equation derived from $L(\vec{\mathbf{y}}) = 0$ is $r^2(r^2+1)^2 = 0$. The correct guess for the particular solution of $L(\vec{\mathbf{y}}) = \cos t + \sin 2t$ is $y_p = At\cos t + Bt \sin t + C\cos 2t + D \sin 2t$.

(e) The phase line for $T' = T^2 - T^4$ is



(f) If the determinant of the $n \times n$ matrix $\mathbf{A}$ is nonzero, then there exist vectors $\vec{\mathbf{x}}$ and $\vec{\mathbf{b}}$ in $\mathbb{R}^n$ making the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ inconsistent.

(g) A tank initially contains 500 liters of water in which 50 grams of salt are dissolved. Pure water enters the tank at 4 liters per minute and the well-mixed solution leaves the tank at 5 liters per minute. The differential equation describing this situation will be homogeneous, linear and autonomous.

SOLUTION:

(a) **FALSE** The zero vector is not in $\mathbb{W}$.

(b) **FALSE** The h-nullcline is $15x_1 - 7x_2 = 1$ which is a line but it does not go through the origin.

(c) **TRUE** The determinant of the matrix in this case vanishes.

(d) **FALSE** $t\cos t$ and $t \sin t$ are solutions to the homogeneous problem. The correct form of the particular solution is $y_p = At^2 \cos t + Bt^2 \sin t + C \cos 2t + D \sin 2t$.

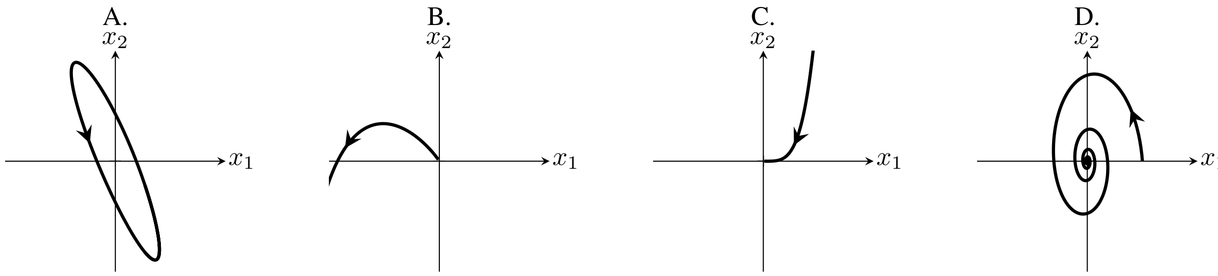
(e) **TRUE** $T' = T^2(1-T)(1+T)$. This is negative for $T > 1$ and $T < -1$ and positive for $-1 < T < 1$.

(f) **FALSE** Since $|\mathbf{A}| \neq 0$, $\vec{\mathbf{x}} = \mathbf{A}^{-1}\vec{\mathbf{b}}$ for all $\vec{\mathbf{b}} \in \mathbb{R}^n$.

(g) **FALSE** The differential equation is $\dfrac{dx}{dt} + \dfrac{x}{500-t} = 0$, which is linear and homogeneous but not autonomous due to the explicit appearance of t.

8. [2360/111622 (20 pts)] Consider the system of differential equations given by $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ for the given matrices below. For each part (a)-(d), (i) compute Tr $\mathbf{A}$, (ii) compute $|\mathbf{A}|$, (iii) state the stability of the fixed point at the origin, (iv) state the geometry of the fixed point at the origin, (v) select the correct phase portrait from those shown in the accompanying figure.

(a) $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix}$ (b) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}$ (c) $\mathbf{A} = \begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix}$ (d) $\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$



SOLUTION:

(a) (i) Tr $\mathbf{A} = 8$ (ii) $|\mathbf{A}| = 16$ (iii) unstable (iv) repelling degenerate node (v) B

(b) (i) Tr $\mathbf{A} = -2$ (ii) $|\mathbf{A}| = 5$ (iii) asymptotically stable (iv) attracting spiral (v) D

(c) (i) Tr $\mathbf{A} = 0$ (ii) $|\mathbf{A}| = 9$ (iii) neutrally stable (iv) center (v) A

(d) (i) Tr $\mathbf{A} = -4$ (ii) $|\mathbf{A}| = 3$ (iii) asymptotically stable (iv) attracting node or node sink (v) C