- This exam is worth 150 points and has 8 problems.
- **Show all work and simplify your answers!** Answers with no justification will receive no points unless otherwise noted.
- **Please begin each problem on a new page.**
- **DO NOT** leave the exam until you have satisfactorily scanned and uploaded your exam to Gradescope.
- You are taking this exam in a proctored and honor code enforced environment. **NO** calculators, cell phones, or other electronic devices or the internet are permitted. You are allowed one 8.5"× 11" crib sheet with writing on two sides.

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0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/111622 (18 pts)] Consider the differential equation $ty'' + (2t-1)y' - 2y = \dfrac{48t^3}{e^{2t}},\ t > 0.$

   (a) (6 pts) Are $y_1 = e^{-2t}$ and $y_2 = 2t - 1$ linearly independent solutions of the associated homogeneous equation? Justify your answer completely.

   (b) (12 pts) Find the general solution of the equation.

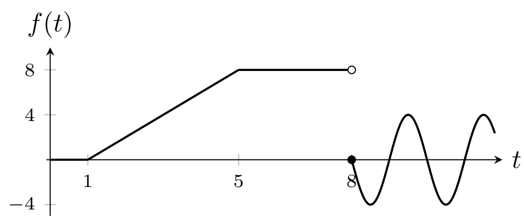
2. [2360/111622 (15 pts)] Solve the initial value problem $\dfrac{y'}{y^2} - \dfrac{3}{xy} = 1,\ y(2) = 2,\ x > 0$ by using the substitution $u(x) = \dfrac{1}{y(x)}$.

3. [2360/111622 (22 pts)] Your niece and nephew have had a lot of fun playing with the harmonic oscillator toy you gave them as a present at the time of the last exam. They have played around with it so much that the damping coefficient and spring/restoring constant have both increased. They have also managed to lose all but two of the stones that came with the toy but the mass of the bucket attached to the spring remains at 1 kg. As a result, the differential equation governing the toy is now $3\ddot{x} + 12\dot{x} + 9x = f(t)$.

   (a) (8 pts) Find the equation of motion if the toy is unforced and initially given an 8 m/s push to the right from a starting position 2 m to the left of the equilibrium position.

   (b) (4 pts) The kids are very interested in knowing the exact time that the bucket will return to the equilibrium position. Provide them (and your grader) an answer.

   (c) (10 pts) To increase their interest in becoming engineers, you have an addition to the toy which provides a driving force, $f(t)$, shown in the following figure. The oscillating portion is given by $-4\sin \pi t$ for $t \geq 8$. The kids (and your grader) want to see the driving force written as a single function using step functions.




4. [2360/111622 (18 pts)] Find the general solution of the following system, writing your answer in the form prescribed by the Nonhomogeneous Principle.

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 3 \\ 2x_1 + 4x_2 - 12x_3 &= 4 \\ 2x_1 + x_2 + 12x_3 &= 7 \\ 3x_1 + 3x_2 + 6x_3 &= 9 \end{aligned}$$

5. [2360/111622 (16 pts)] Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 5 & 0 & 0 \end{bmatrix}$.

   (a) (8 pts) Find the eigenvalues and eigenvectors of $\mathbf{A}$.

   (b) (8 pts) Solve $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}},\ \vec{\mathbf{x}}(0) = \begin{bmatrix} 2 & -1 & -5 \end{bmatrix}^{\mathrm{T}}$. Write your answer as a single vector.

**MORE PROBLEMS AND LAPLACE TRANSFORM TABLE ON THE BACK**

6. [2360/111622 (20 pts)] Solve $y'' + 4y' + 6y = \delta(t-2)$ $y(0) = 2,\ y'(0) = -1$.

7. [2360/111622 (21 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

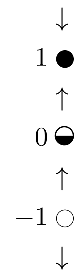
   (a) The set $\mathbb{W}$ of all solutions to $2x_1 - 5x_2 + 7x_3 - 1 = 0$ forms a 2-dimensional subspace of $\mathbb{R}^3$.

   (b) The $h$- and $v$-nullclines of the system $\vec{\mathbf{x}}' = \begin{bmatrix} 2 & -2 \\ 15 & -7 \end{bmatrix} \vec{\mathbf{x}} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are lines through the origin.

   (c) All $n \times n$ matrices that have two identical columns are singular.

   (d) $L(\vec{\mathbf{y}})$ represents a constant coefficient differential operator. The characteristic equation derived from $L(\vec{\mathbf{y}}) = 0$ is $r^2(r^2+1)^2 = 0$. The correct guess for the particular solution of $L(\vec{\mathbf{y}}) = \cos t + \sin 2t$ is $y_p = At\cos t + Bt\sin t + C\cos 2t + D\sin 2t$.

   (e) The phase line for $T' = T^2 - T^4$ is

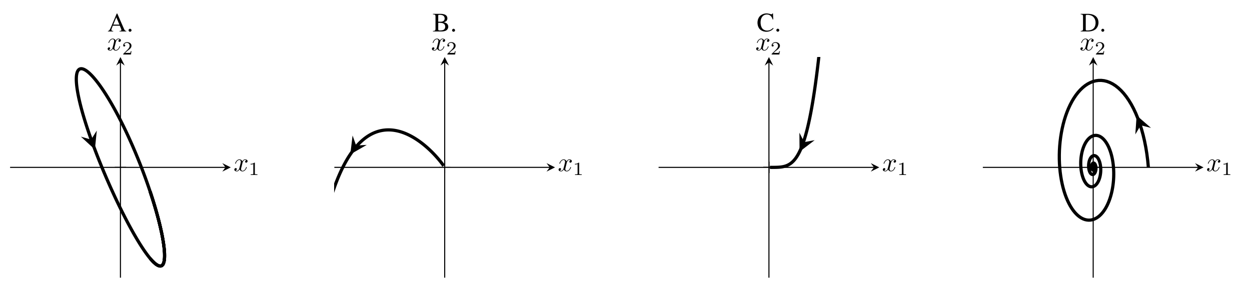



   (f) If the determinant of the $n \times n$ matrix $\mathbf{A}$ is nonzero, then there exist vectors $\vec{\mathbf{x}}$ and $\vec{\mathbf{b}}$ in $\mathbb{R}^n$ making the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ inconsistent.

   (g) A tank initially contains 500 liters of water in which 50 grams of salt are dissolved. Pure water enters the tank at 4 liters per minute and the well-mixed solution leaves the tank at 5 liters per minute. The differential equation describing this situation will be homogeneous, linear and autonomous.

8. [2360/111622 (20 pts)] Consider the system of differential equations given by $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ for the given matrices below. For each part (a)-(d), (i) compute Tr $\mathbf{A}$, (ii) compute $|\mathbf{A}|$, (iii) state the stability of the fixed point at the origin, (iv) state the geometry of the fixed point at the origin, (v) select the correct phase portrait from those shown in the accompanying figure.

   (a) $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix}$  (b) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}$  (c) $\mathbf{A} = \begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix}$  (d) $\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

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**Short table of Laplace Transforms:** $\mathscr{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, $a, b, c$ are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

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$$\mathscr{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\{e^{at} f(t)\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{tf'(t)\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\{f(t-c)\,\text{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\text{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$