1. [2360/111622 (10 pts)] Consider the differential equation $\dfrac{\mathrm{d}^2u}{\mathrm{d}t^2} - \dfrac{2}{t}\dfrac{\mathrm{d}u}{\mathrm{d}t} + \dfrac{2+t^2}{t^2}u = 0$ for $t > 0$.

   (a) (4 pts) Show that $u_1 = t\sin t$ is a solution to the differential equation.

   (b) (6 pts) Assuming that $u_2 = t\cos t$ is also a solution to the differential equation, is the differential equation's solution space, $\mathbb{S}$, equal to span$\{u_1, u_2\}$? Justify your answer.

SOLUTION:

(a)

$$u_1' = t\cos t + \sin t$$

$$u_1'' = -t\sin t + 2\cos t$$

$$u_1'' - \frac{2}{t}u_1' + \frac{2+t^2}{t^2}u_1 = -t\sin t + 2\cos t - \frac{2}{t}\left(t\cos t + \sin t\right) + \left(\frac{2+t^2}{t^2}\right)(t\sin t)$$

$$= -t\sin t + 2\cos t - 2\cos t - \frac{2\sin t}{t} + \frac{2\sin t}{t} + t\sin t = 0$$

(b) Since this is a second order linear homogeneous equation, the dimension of the solution space is two. u_1, u_2 are both solutions. If, furthermore, u_1 and u_2 are linearly independent they form a basis for the solution space and $\mathbb{S} = \text{span}\left\{t\sin t, t\cos t\right\}$.

$$W[t\sin t, t\cos t] = \begin{vmatrix} t\sin t & t\cos t \\ t\cos t + \sin t & -t\sin t + \cos t \end{vmatrix}$$

$$= -t^2\sin^2 t + t\sin t\cos t - (t^2\cos^2 t + t\sin t\cos t) = -t^2 \neq 0$$

Since the Wronskian is not zero, the solutions are linearly independent and thus form a basis for $\mathbb{S}$.

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2. [2360/111622 (18 pts)] Let $L(\vec{\mathbf{y}}) = y'' + \dfrac{1}{t}y' - \dfrac{1}{t^2}y$. A basis for the solution space of $L(\vec{\mathbf{y}}) = 0$ is $\left\{2t, t^{-1}\right\}$. Find the general solution of $L(\vec{\mathbf{y}}) = \dfrac{1}{t^2}$.

SOLUTION:

Since this is a variable coefficient problem, we must use variation of parameters with $f(t) = t^{-2}$ and $y_1 = 2t$ and $y_2 = t^{-1}$ to find a particular solution of the form $y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$.

$$W[y_1, y_2] = \begin{vmatrix} 2t & t^{-1} \\ 2 & -t^{-2} \end{vmatrix} = -4t^{-1}$$

$$v_1' = -\frac{y_2(t)f(t)}{W[y_1,y_2]} = \frac{(-t^{-1})(t^{-2})}{-4t^{-1}} = \frac{1}{4}t^{-2} \implies v_1 = \frac{1}{4}\int t^{-2}\,\mathrm{d}t = -\frac{1}{4t}$$

$$v_2' = \frac{y_1(t)f(t)}{W[y_1,y_2]} = \frac{2t(t^{-2})}{-4t^{-1}} = -\frac{1}{2} \implies v_2 = -\int \frac{1}{2}\,\mathrm{d}t = -\frac{t}{2}$$

$$y_p = -\frac{1}{4t}(2t) - \frac{t}{2}t^{-1} = -\frac{1}{2} - \frac{1}{2} = -1$$

The general solution is $y = y_h + y_p = c_1 t + \dfrac{c_2}{t} - 1$ ■

3. [2360/111622 (24 pts)] With the upcoming holidays, you have decided to give your niece and nephew a sweet new toy. It consists of a 1-kg bucket hooked to a spring which is then attached to a wall. The restoring/spring constant is $\frac{1}{4}$ and the device is hooked up so that the damping force is equal to 4 times the instantaneous velocity of the bucket. A bag of 1-kg stones is included with the toy. Hint: There are no irrational numbers involved in this problem.

   (a) (20 pts) Upon seeing the toy, the kids put 6 stones into the 1-kg bucket, pull the bucket 1 meter to the right of the equilibrium position and push it to the left at one-half meter per second. How many times will the bucket pass through the equilibrium position?

(b) (2 pts) What is the smallest number of stones that need to be in the bucket so that the kids can have the excitement of watching the bucket oscillate?

(c) (2 pts) The kids really want a thrill and ask you to try to make the toy exhibit resonance. So you have them put 15 stones into the bucket and place the toy onto a table which provides a driving force to the toy given by $f(t) = \cos \frac{1}{8}t$. Will the kids get a thrill?

SOLUTION:

(a) The initial value problem governing the motion is $7\ddot{x} + 4\dot{x} + \dfrac{1}{4}x = 0,\ x(0) = 1,\ \dot{x}(0) = -\dfrac{1}{2}$. The characteristic equation gives

$$7r^2 + 4r + \frac{1}{4} = 0$$

$$r = \frac{-4 \pm \sqrt{4^2 - 4(7)\left(\frac{1}{4}\right)}}{2(7)} = \frac{-4 \pm 3}{14} = -\frac{1}{2}, -\frac{1}{14}$$

so the general solution is $x(t) = c_1 e^{-t/2} + c_2 e^{-t/14}$. Applying the initial conditions yields the linear system

$$c_1 + c_2 = 1$$

$$-\frac{1}{2}c_1 - \frac{1}{14}c_2 = -\frac{1}{2}$$

$$c_1 = \frac{\begin{vmatrix} 1 & 1 \\ -\frac{1}{2} & -\frac{1}{14} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -\frac{1}{2} & -\frac{1}{14} \end{vmatrix}} = 1 \qquad c_2 = \frac{\begin{vmatrix} 1 & 1 \\ -\frac{1}{2} & -\frac{1}{2} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -\frac{1}{2} & -\frac{1}{14} \end{vmatrix}} = 0$$

So the solution to the initial value problem is $x(t) = e^{-t/2}$ which is never zero, implying that the bucket never passes through the equilibrium position.

(b) We need $b^2 - 4mk = 16 - 4(m)\left(\dfrac{1}{4}\right) = 16 - m < 0$ which will occur first here when $m = 17$. At least sixteen stones need to be in the bucket since the bucket itself has a mass of 1 kg.

(c) No. Even though $\omega_0 = \frac{1}{8}$ matches the forcing frequency, damped oscillators never experience resonance.

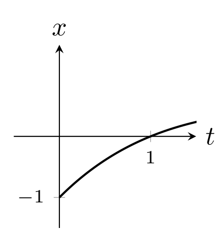
4. [2360/111622 (8 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

(a) The differential equation $3\ddot{u} + 2\cos u = 5$ describes a conservative system.

(b) The differential equation $y''' - y = 1$ can be written as a system of differential equations in the form $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ where $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

(c) The solution to a certain differential equation is $x(t) = e^{-t} + 2te^{-t} + e^{-2t}\cos 4t + \sin 5t$. The steady state portion of the solution is $e^{-2t}\cos 4t + \sin 5t$.

(d) A portion of the equation of motion for an unforced mass/spring system is shown in the following figure. From the graph, the mass was released from rest 1 unit to the left of its equilibrium position.



SOLUTION:

(a) **TRUE** It can be written in the form $m\ddot{u} + V'(u) = 0$ where $m = 3$ and $V'(u) = 2\cos u - 5$

(b) **FALSE** Let $x_1 = y, x_2 = y', x_3 = y''$. Then

$$x_1' = y' = x_2$$
$$x_2' = y'' = x_3$$
$$x_3' = y''' = 1 + y = 1 + x_1$$

With $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ we have

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}} + \vec{\mathbf{f}}$$

(c) **FALSE** The steady state solution is $\sin 5t$.

(d) **FALSE** The slope of the graph at $t = 0$ is positive meaning the initial velocity was positive, not zero.

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5. [2360/111622 (20 pts)] Use the Method of Undetermined Coefficients to find the general solution of $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t} = -40\cos^2 t$. Hint: $2\cos^2 t = 1 + \cos 2t$.

SOLUTION:

Using the hint, rewrite the equation as $y'' - y' = -20(2\cos^2 t) = -20(1 + \cos 2t) = -20 - 20\cos 2t$. Find the solution to the associated homogeneous equation.

$$y_h'' - y_h' = 0$$
$$r^2 - r = r(r-1) = 0$$
$$r = 0, 1$$
$$y_h = c_1 + c_2 e^t$$

Find the particular solution. Note that a constant is a solution to the associated homogeneous equation.

$$y_p = At + B\cos 2t + C\sin 2t$$
$$y_p' = A - 2B\sin 2t + 2C\cos 2t$$
$$y_p'' = -4B\cos 2t - 4C\sin 2t$$

$$\begin{aligned} y_p'' - y_p' &= -4B\cos 2t - 4C\sin 2t - (A - 2B\sin 2t + 2C\cos 2t) \\ &= -A + (-4B - 2C)\cos 2t + (2B - 4C)\sin 2t = -20 - 20\cos 2t \end{aligned}$$

$$-A = -20 \implies A = 20$$
$$-4B - 2C = -20 \implies 2B + C = 10$$
$$2B - 4C = 0 \implies B - 2C = 0$$

$$B = \frac{\begin{vmatrix} 10 & 1 \\ 0 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{-20}{-5} = 4 \qquad C = \frac{\begin{vmatrix} 2 & 10 \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{-10}{-5} = 2$$

$$y = y_h + y_p = c_1 + c_2 e^t + 20t + 4\cos 2t + 2\sin 2t$$

6. [2360/111622 (20 pts)] Use Laplace Transforms to solve the initial value problem $y'' + 2y' = 12e^{-2t},\ y(0) = 5,\ y'(0) = 0$. Hint: The following may be handy: $\dfrac{1}{s(s+2)^2} = \dfrac{1}{4}\left[\dfrac{1}{s} - \dfrac{1}{s+2} - \dfrac{2}{(s+2)^2}\right]$

SOLUTION:

$$\mathscr{L}\left\{y'' + 2y' = 12e^{-2t}\right\}$$

$$s^2Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] = \frac{12}{s+2}$$

$$[s(s+2)]Y(s) = \frac{12}{s+2} + 5(s+2)$$

$$Y(s) = \frac{12}{s(s+2)^2} + \frac{5}{s}$$

$$y(t) = \mathscr{L}^{-1}\left\{\frac{12}{s(s+2)^2} + \frac{5}{s}\right\} = \mathscr{L}^{-1}\left\{3\left(\frac{1}{s} - \frac{1}{s+2} - \frac{2}{(s+2)^2}\right) + \frac{5}{s}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{8}{s}\right\} - 3\mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\} - 6\mathscr{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} = 8 - 3e^{-2t} - 6te^{-2t}$$

$$= 8 - 3e^{-2t}(2t+1)$$