- This exam is worth 100 points and has 6 problems.
- **Show all work and simplify your answers!** Answers with no justification will receive no points unless otherwise noted.
- **Please begin each problem on a new page.**
- **DO NOT** leave the exam until you have satisfactorily scanned and uploaded your exam to Gradescope.
- You are taking this exam in a proctored and honor code enforced environment. **NO** calculators, cell phones, or other electronic devices or the internet are permitted. You are allowed one 8.5"× 11" crib sheet with writing on one side.

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0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/111622 (10 pts)] Consider the differential equation $\dfrac{\mathrm{d}^2u}{\mathrm{d}t^2} - \dfrac{2}{t}\dfrac{\mathrm{d}u}{\mathrm{d}t} + \dfrac{2+t^2}{t^2}u = 0$ for $t > 0$.

   (a) (4 pts) Show that $u_1 = t\sin t$ is a solution to the differential equation.

   (b) (6 pts) Assuming that $u_2 = t\cos t$ is also a solution to the differential equation, is the differential equation's solution space, $\mathbb{S}$, equal to span$\{u_1, u_2\}$? Justify your answer.

2. [2360/111622 (18 pts)] Let $L(\vec{\mathbf{y}}) = y'' + \frac{1}{t}y' - \frac{1}{t^2}y$. A basis for the solution space of $L(\vec{\mathbf{y}}) = 0$ is $\{2t, t^{-1}\}$. Find the general solution of $L(\vec{\mathbf{y}}) = \dfrac{1}{t^2}$.

3. [2360/111622 (24 pts)] With the upcoming holidays, you have decided to give your niece and nephew a sweet new toy. It consists of a 1-kg bucket hooked to a spring which is then attached to a wall. The restoring/spring constant is $\frac{1}{4}$ and the device is hooked up so that the damping force is equal to 4 times the instantaneous velocity of the bucket. A bag of 1-kg stones is included with the toy. Hint: There are no irrational numbers involved in this problem.

   (a) (20 pts) Upon seeing the toy, the kids put 6 stones into the 1-kg bucket, pull the bucket 1 meter to the right of the equilibrium position and push it to the left at one-half meter per second. How many times will the bucket pass through the equilibrium position?

   (b) (2 pts) What is the smallest number of stones that need to be in the bucket so that the kids can have the excitement of watching the bucket oscillate?

   (c) (2 pts) The kids really want a thrill and ask you to try to make the toy exhibit resonance. So you have them put 15 stones into the bucket and place the toy onto a table which provides a driving force to the toy given by $f(t) = \cos\frac{1}{8}t$. Will the kids get a thrill?

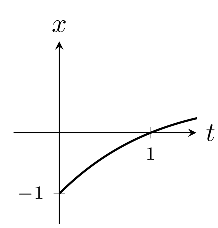
4. [2360/111622 (8 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) The differential equation $3\ddot{u} + 2\cos u = 5$ describes a conservative system.

   (b) The differential equation $y''' - y = 1$ can be written as a system of differential equations in the form $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ where $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

   (c) The solution to a certain differential equation is $x(t) = e^{-t} + 2te^{-t} + e^{-2t}\cos 4t + \sin 5t$. The steady state portion of the solution is $e^{-2t}\cos 4t + \sin 5t$.

   (d) A portion of the equation of motion for an unforced mass/spring system is shown in the following figure. From the graph, the mass was released from rest 1 unit to the left of its equilibrium position.



MORE PROBLEMS AND LAPLACE TRANSFORM TABLE ON THE BACK

5. [2360/111622 (20 pts)] Use the Method of Undetermined Coefficients to find the general solution of $\dfrac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \dfrac{\mathrm{d}y}{\mathrm{d}t} = -40\cos^2 t$. Hint: $2\cos^2 t = 1 + \cos 2t$.

6. [2360/111622 (20 pts)] Use Laplace Transforms to solve the initial value problem $y'' + 2y' = 12e^{-2t},\ y(0) = 5,\ y'(0) = 0$. Hint: The following may be handy: $\dfrac{1}{s(s+2)^2} = \dfrac{1}{4}\left[\dfrac{1}{s} - \dfrac{1}{s+2} - \dfrac{2}{(s+2)^2}\right]$

Short table of Laplace Transforms: $\mathscr{L}\{f(t)\} = F(s) \equiv \displaystyle\int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a, b, c are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{tf'(t)\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\{f(t-c)\,\mathrm{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\mathrm{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$
