1. [2360/101922 (24 pts)] The following parts are not related.

   (a) (10 pts) Show that the Wronskian of the functions \( \{ t + 3, t^2 - 1, 2t^2 - t - 5 \} \) cannot be used to determine if the functions are linearly dependent or independent.

   (b) (14 pts) Consider the linear system
   
   \[
   \begin{align*}
   x_1 + 2x_2 + x_3 &= 4 \\
   x_1 + 2x_2 &= 2 \\
   x_1 + x_2 + 3x_3 &= -7 
   \end{align*}
   \]

   Use the inverse of the coefficient matrix to find the solution to the system. You must use elementary row operations to find the inverse.

   **Solution:**

   (a)
   
   \[
   W[t + 3, t^2 - 1, 2t^2 - t - 5](t) = \begin{vmatrix} t + 3 & t^2 - 1 & 2t^2 - t - 5 \\ 1 & 2t & 4t - 1 \\ 0 & 2 & 4 \end{vmatrix}
   \]
   
   \[
   = 2(-1)^{3+2} \begin{vmatrix} t + 3 & 2t^2 - t - 5 \\ 1 & 4t - 1 \end{vmatrix} + 4(-1)^{3+3} \begin{vmatrix} t + 3 & t^2 - 1 \\ 1 & 2t \end{vmatrix}
   \]
   
   \[
   = -2(4t^2 - t + 12t - 3 - 2t^2 + t + 5) + 4(2t^2 + 6t - t^2 + 1)
   \]
   
   \[
   = -4t^2 - 24t - 4 + 4t^2 + 24t + 4 = 0
   \]

   Since the Wronskian of the functions vanishes identically, we can conclude nothing about the linear dependence or independence of the functions.

   (b) The system can be written as \( A\vec{x} = \vec{b} \) with \( A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{bmatrix}, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \vec{b} = \begin{bmatrix} 4 \\ 2 \\ -7 \end{bmatrix} \). We need to find \( A^{-1} \) so that

   \[
   \vec{x} = A^{-1}\vec{b}.
   \]

   \[
   \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
   \]

   \[
   \Rightarrow A^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -6 & 5 & 2 \\ 3 & -2 & -1 \\ 1 & -1 & 0 \end{bmatrix}
   \]

   \[
   \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6 & 5 & 2 \\ 3 & -2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ -28 \\ 15 \end{bmatrix} = \begin{bmatrix} 2 \\ -14 \\ 15 \end{bmatrix}
   \]

2. [2360/101922 (20 pts)] The following parts are not related. Justify your answers.

   (a) (10 pts) Does the set \( \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\} \) form a basis for \( \mathbb{R}^3 \)?

   (b) (10 pts) Determine if the following subsets \( \mathbb{W} \) of the given vector space \( \mathbb{V} \) are subspaces. Assume that the standard definitions of vector addition and scalar multiplication apply.

   i. (5 pts) \( \mathbb{V} = \mathbb{M}_{33}; \) \( \mathbb{W} \) is the set of matrices of the form \( \begin{bmatrix} r & 0 & p \\ 0 & s & 0 \\ q & 0 & t \end{bmatrix} \) where \( p, q \) are real numbers and \( r, s, t \) are integers.

   ii. (5 pts) \( \mathbb{V} = \mathbb{R}^3; \) \( \mathbb{W} \) is the set of solutions to the equation \( 4x_1 - 3x_2 + 9x_3 = 0 \).
SOLUTION:

(a) There are three vectors in \( \mathbb{R}^3 \), a vector space of dimension 3. Check for linear independence:

\[
\begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix} + \begin{bmatrix}
2 \\
-1 \\
1
\end{bmatrix} = \begin{bmatrix}
3 \\
0 \\
0
\end{bmatrix} \iff \begin{bmatrix}
1 & 0 & 2 \\
1 & 1 & 1 \\
0 & 1 & -1
\end{bmatrix} \begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix} = \begin{bmatrix} 0 \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 \\
1 & 1 \\
0 & 1
\end{bmatrix} = 1(-1)^{1+1} \begin{bmatrix} 1 & 1 \\
1 & -1 \\
0 & 1
\end{bmatrix} + 1(-1)^{2+1} \begin{bmatrix} 0 & 2 \\
1 & 1 \\
1 & -1
\end{bmatrix} = -2 + 2 = 0
\]

The vectors are linearly dependent and thus cannot form a basis for \( \mathbb{R}^3 \).

(b) i. \( \mathcal{W} \) is not a subspace. \( \mathcal{W} \) is closed under vector addition (the sum of two integers is an integer) but is not closed under scalar multiplication. For example,

\[
\vec{u} = \mathbf{I} \in \mathcal{W} \text{ but } \sqrt{2}\vec{u} = \begin{bmatrix} \sqrt{2} \\
0 \\
0 \end{bmatrix} \notin \mathcal{W}
\]

ii. \( \mathcal{W} \) is a subspace. Use linear combinations to check for closure. Let \( \alpha, \beta \in \mathbb{R} \) and

\[
\vec{u} = \begin{bmatrix} u_1 \\
u_2 \\
u_3 \end{bmatrix} \in \mathcal{W} \implies 4u_1 - 3u_2 + 9u_3 = 0 \text{ and } \vec{v} = \begin{bmatrix} v_1 \\
v_2 \\
v_3 \end{bmatrix} \in \mathcal{W} \implies 4v_1 - 3v_2 + 9v_3 = 0
\]

Then \( \alpha \vec{u} + \beta \vec{v} = \begin{bmatrix} \alpha u_1 + \beta v_1 \\
\alpha u_2 + \beta v_2 \\
\alpha u_3 + \beta v_3 \end{bmatrix} \) and

\[
4(\alpha u_1 + \beta v_1) - 3(\alpha u_2 + \beta v_2) + 9(\alpha u_3 + \beta v_3) = \alpha(4u_1 - 3u_2 + 9u_3) + \beta(4v_1 - 3v_2 + 9v_3) = \alpha(0) + \beta(0) = 0 \implies \alpha \vec{u} + \beta \vec{v} \in \mathcal{W}
\]

\( \mathcal{W} \) is closed under vector addition and scalar multiplication and therefore a subspace. Note also that the equation is linear and homogeneous and hence the Superposition Principle applies, which is essentially the closure properties.

3. [2360/101922 (12 pts)] Write the word TRUE or FALSE as appropriate. No work need be shown. No partial credit given.

(a) For any matrix \( A \), \( |A^T A| \) and \( |AA^T| \) are defined.
(b) For invertible matrices \( A \) and \( B \), \(|AB^{-1}| = |A||B|\).
(c) If square matrix \( A \) has 0 for an eigenvalue, then Cramer’s Rule can be used to solve the system \( AX = B \).
(d) \( \text{Tr} \[(2I)^3]\) = 24 where \( I \) is the 3 \( \times \) 3 identity matrix.
(e) \( \begin{bmatrix} 1 & 2 \\
3 & 4 \end{bmatrix} \in \text{span} \left\{ \begin{bmatrix} 1 & 0 \\
0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\
0 & 0 \end{bmatrix} \right\} \)
(f) Homogeneous systems of linear algebraic equations are never inconsistent.

SOLUTION:

(a) TRUE If \( A \) is \( m \times n \), then \( A^T A \) is \( n \times n \) and \( AA^T \) is \( m \times m \) which are both square and the determinant is defined.
(b) TRUE \( |AB^{-1}| = |A||B|^{-1} = |A||B| \)
(c) FALSE If 0 is an eigenvalue of a square matrix, then \( |A| = 0 \) and Cramer’s Rule cannot be used.
(d) TRUE \( \text{Tr} \[(2I)^3]\) = \( \text{Tr} \left[ \begin{bmatrix} 2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2 \end{bmatrix}^3 \right] = \text{Tr} \left[ \begin{bmatrix} 2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2 \end{bmatrix} \right] = \text{Tr} \begin{bmatrix} 8 & 0 & 0 \\
0 & 8 & 0 \\
0 & 0 & 8 \end{bmatrix} = 24 \)
(e) FALSE \( c_1 \begin{bmatrix} 1 & 0 \\
0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\
0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\
3 & 4 \end{bmatrix} \) has no solution. \( 0c_1 + 0c_2 \) can never equal 3.
4. [2360/101922 (22 pts)] Consider the augmented matrix
\[
\begin{bmatrix}
1 & 0 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]
and the system \( A \vec{x} = \vec{0} \).

(a) (10 pts) If the matrix is in RREF, write the word YES. Otherwise, write NO and put the matrix in RREF.
(b) (12 pts) Find a basis for the solution space of the system. What is the dimension of the solution space?

\textbf{SOLUTION:}

(a) NO. The following operations will put the matrix into RREF: \( R_2 \leftrightarrow R_3 \) then \( R_1^* = -1R_2 + R_1 \).

(b) From the RREF, the leading/basic variables corresponding to the pivot columns are \( x_1 \) and \( x_4 \). The free variables are \( x_2 = s \) and \( x_3 = t \). The solutions of the system are
\[
\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ s \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad s,t \in \mathbb{R}
\]
or equivalently \( \vec{x} \in \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \).

The solution space has dimension 2 with basis \( \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \).

5. [2360/101922 (22 pts)] The following parts are not related.

(a) (10 pts) The characteristic polynomial of a certain matrix \( B \) is \( p(\lambda) = 2\lambda^6 + 6\lambda^5 + 5\lambda^4 \).

i. (2 pts) What is the order/size of the matrix \( B \)?

ii. (8 pts) Find the eigenvalues of \( B \) and their (algebraic) multiplicities. Do not find the eigenvectors.

(b) (12 pts) Let \( A = \begin{bmatrix} -7 & 4 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 4 & -7 & -1 \\ 0 & 0 & 0 & -7 \end{bmatrix} \) and let \( E_{-7} \) denote the eigenspace associated with eigenvalue \( \lambda = -7 \). Find a basis for \( E_{-7} \) and the dimension of \( E_{-7} \).

\textbf{SOLUTION:}

(a) i. Since the characteristic polynomial is degree 6, \( B \) is of order 6 or \( 6 \times 6 \).

ii. \[
2\lambda^6 + 6\lambda^5 + 5\lambda^4 = 0
\]
\[
\lambda^4(2\lambda^2 + 6\lambda + 5) = 0 \quad \Rightarrow \quad \lambda = 0 \quad \text{or} \quad 2\lambda^2 + 6\lambda + 5 = 0
\]
\[
\lambda = \frac{-6 \pm \sqrt{6^2 - 4(2)(5)}}{2} = \frac{-6 \pm \sqrt{-4}}{4} = \frac{-6 \pm 2i}{4} = \frac{-3 \pm i}{2}
\]
The eigenvalues of \( B \) are 0 with multiplicity of 4 and \( \frac{-3 + i}{2} \), \( \frac{-3 - i}{2} \), each with multiplicity of 1.

(b) We need to solve \( (A + 7I) \vec{v} = \vec{0} \)

\[
\begin{bmatrix}
0 & 4 & 0 & -1 \\
0 & 8 & 0 & -2 \\
0 & 4 & 0 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
R_2 = -2R_1 + R_2 \\
R_3 = -1R_1 + R_3 \\
R_1^* = \frac{1}{4}R_1
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 & -\frac{1}{4} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
\[
\Rightarrow
\begin{bmatrix}
\lambda_1 = r \\
\lambda_2 = \frac{1}{4}r \\
\lambda_3 = s \\
\lambda_4 = t
\end{bmatrix}
\]
A basis for $E_7$ is \[
\begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
0 \\
1 \\
0 \\
0 \\
1 \\
4
\end{bmatrix}, \begin{bmatrix}
0 \\
1 \\
0 \\
0 \\
0 \\
0 \\
4
\end{bmatrix}
\] which has dimension 3.