1. [2360/101922 (24 pts)] The following parts are not related.

   (a) (10 pts) Show that the Wronskian of the functions $\{t+3, t^2-1, 2t^2-t-5\}$ cannot be used to determine if the functions are linearly dependent or independent.

   (b) (14 pts) Consider the linear system

$$x_1 + 2x_2 + x_3 = 4$$
$$x_1 + 2x_2 = 2$$
$$x_1 + x_2 + 3x_3 = -7$$

   Use the inverse of the coefficient matrix to find the solution to the system. You must use elementary row operations to find the inverse.

SOLUTION:

(a)

$$W[t+3, t^2-1, 2t^2-t-5](t) = \begin{vmatrix} t+3 & t^2-1 & 2t^2-t-5 \\ 1 & 2t & 4t-1 \\ 0 & 2 & 4 \end{vmatrix}$$
$$= 2(-1)^{3+2}\begin{vmatrix} t+3 & 2t^2-t-5 \\ 1 & 4t-1 \end{vmatrix} + 4(-1)^{3+3}\begin{vmatrix} t+3 & t^2-1 \\ 1 & 2t \end{vmatrix}$$
$$= -2(4t^2-t+12t-3-2t^2+t+5) + 4(2t^2+6t-t^2+1)$$
$$= -4t^2-24t-4+4t^2+24t+4 = 0$$

Since the Wronskian of the functions vanishes identically, we can conclude nothing about the linear dependence or independence of the functions.

(b) The system can be written as $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ with $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}$, $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\vec{\mathbf{b}} = \begin{bmatrix} 4 \\ 2 \\ -7 \end{bmatrix}$. We need to find $\mathbf{A}^{-1}$ so that $\vec{\mathbf{x}} = \mathbf{A}^{-1}\vec{\mathbf{b}}$.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -3 & 2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & 5 & 2 \\ 0 & 1 & 0 & 3 & -2 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right] \implies \mathbf{A}^{-1} = \begin{bmatrix} -6 & 5 & 2 \\ 3 & -2 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6 & 5 & 2 \\ 3 & -2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ -7 \end{bmatrix} = \begin{bmatrix} -28 \\ 15 \\ 2 \end{bmatrix}$$

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2. [2360/101922 (20 pts)] The following parts are not related. Justify your answers.

   (a) (10 pts) Does the set $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}\right\}$ form a basis for $\mathbb{R}^3$?

   (b) (10 pts) Determine if the following subsets $\mathbb{W}$ of the given vector space $\mathbb{V}$ are subspaces. Assume that the standard definitions of vector addition and scalar multiplication apply.

      i. (5 pts) $\mathbb{V} = \mathbb{M}_{33}$; $\mathbb{W}$ is the set of matrices of the form $\begin{bmatrix} r & 0 & p \\ 0 & s & 0 \\ q & 0 & t \end{bmatrix}$ where p, q are real numbers and r, s, t are integers.

      ii. (5 pts) $\mathbb{V} = \mathbb{R}^3$; $\mathbb{W}$ is the set of solutions to the equation $4x_1 - 3x_2 + 9x_3 = 0$.

SOLUTION:

(a) There are three vectors in $\mathbb{R}^3$, a vector space of dimension 3. Check for linear independence:

$$c_1 \begin{bmatrix}1\\1\\0\end{bmatrix} + c_2 \begin{bmatrix}0\\1\\1\end{bmatrix} + c_3 \begin{bmatrix}2\\1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \iff \begin{bmatrix}1&0&2\\1&1&1\\0&1&-1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{vmatrix}1&0&2\\1&1&1\\0&1&-1\end{vmatrix} = 1(-1)^{1+1}\begin{vmatrix}1&1\\1&-1\end{vmatrix} + 1(-1)^{2+1}\begin{vmatrix}0&2\\1&-1\end{vmatrix} = -2+2=0$$

The vectors are linearly dependent and thus cannot form a basis for $\mathbb{R}^3$.

(b) i. $\mathbb{W}$ is not a subspace. $\mathbb{W}$ is closed under vector addition (the sum of two integers is an integer) but is not closed under scalar multiplication. For example,

$$\vec{\mathbf{u}} = \mathbf{I} \in \mathbb{W} \quad \text{but} \quad \sqrt{2}\vec{\mathbf{u}} = \begin{bmatrix}\sqrt{2}&0&0\\0&\sqrt{2}&0\\0&0&\sqrt{2}\end{bmatrix} \notin \mathbb{W}$$

ii. $\mathbb{W}$ is a subspace. Use linear combinations to check for closure. Let $\alpha, \beta \in \mathbb{R}$ and

$$\vec{\mathbf{u}} = \begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} \in \mathbb{W} \implies 4u_1 - 3u_2 + 9u_3 = 0 \quad \text{and} \quad \vec{\mathbf{v}} = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} \in \mathbb{W} \implies 4v_1 - 3v_2 + 9v_3 = 0$$

Then $\alpha\vec{\mathbf{u}} + \beta\vec{\mathbf{v}} = \begin{bmatrix}\alpha u_1 + \beta v_1\\ \alpha u_2 + \beta v_2\\ \alpha u_3 + \beta v_3\end{bmatrix}$ and

$$\begin{aligned}4(\alpha u_1 + \beta v_1) - 3(\alpha u_2 + \beta v_2) + 9(\alpha u_3 + \beta u_3)&\\ &= \alpha(4u_1 - 3u_2 + 9u_3) + \beta(4v_1 - 3v_2 + 9v_3)\\ &= \alpha(0) + \beta(0)\\ &= 0+0=0 \implies \alpha\vec{\mathbf{u}} + \beta\vec{\mathbf{v}} \in \mathbb{W}\end{aligned}$$

$\mathbb{W}$ is closed under vector addition and scalar multiplication and therefore a subspace. Note also that the equation is linear and homogeneous and hence the Superposition Principle applies, which is essentially the closure properties.

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3. [2360/101922 (12 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

(a) For any matrix $\mathbf{A}$, $\left|\mathbf{A}^{\mathrm{T}}\mathbf{A}\right|$ and $\left|\mathbf{A}\mathbf{A}^{\mathrm{T}}\right|$ are defined.

(b) For invertible matrices $\mathbf{A}$ and $\mathbf{B}$, $\left|\mathbf{A}\mathbf{B}^{-1}\right| = |\mathbf{A}|/|\mathbf{B}|$.

(c) If square matrix $\mathbf{A}$ has 0 for an eigenvalue, then Cramer's Rule can be used to solve the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

(d) Tr $\left[(2\mathbf{I})^3\right] = 24$ where $\mathbf{I}$ is the 3×3 identity matrix.

(e) $\begin{bmatrix}1&2\\3&4\end{bmatrix} \in \operatorname{span}\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}\right\}$

(f) Homogeneous systems of linear algebraic equations are never inconsistent.

SOLUTION:

(a) **TRUE** If $\mathbf{A}$ is $m \times n$, then $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is $n \times n$ and $\mathbf{A}\mathbf{A}^{\mathrm{T}}$ is $m \times m$ which are both square and the determinant is defined.

(b) **TRUE** $|\mathbf{A}\mathbf{B}^{-1}| = |\mathbf{A}||\mathbf{B}^{-1}| = |\mathbf{A}||\mathbf{B}|^{-1} = |\mathbf{A}|/|\mathbf{B}|$

(c) **FALSE** If 0 is an eigenvalue of a square matrix, then $|\mathbf{A}| = 0$ and Cramer's Rule cannot be used.

(d) **TRUE** $\operatorname{Tr}\left[(2\mathbf{I})^3\right] = \operatorname{Tr}\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}^3 = \operatorname{Tr}\left(\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}\right) = \operatorname{Tr}\begin{bmatrix}8&0&0\\0&8&0\\0&0&8\end{bmatrix} = 24$

(e) **FALSE** $c_1\begin{bmatrix}1&0\\0&1\end{bmatrix} + c_2\begin{bmatrix}0&1\\0&0\end{bmatrix} = \begin{bmatrix}1&2\\3&4\end{bmatrix}$ has no solution. $0c_1 + 0c_2$ can never equal 3.

(f) **TRUE** They always have at least the trivial solution.

4. [2360/101922 (22 pts)] Consider the augmented matrix $\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$ derived from the linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

(a) (10 pts) If the matrix is in RREF, write the word YES. Otherwise, write NO and put the matrix in RREF.

(b) (12 pts) Find a basis for the solution space of the system. What is the dimension of the solution space?

SOLUTION:

(a) NO. The following operations will put the matrix into RREF: $R_2 \leftrightarrow R_3$ then $R_1^* = -1R_2 + R_1$.

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(b) From the RREF, the leading/basic variables corresponding to the pivot columns are x_1 and x_4. The free variables are $x_2 = s$ and $x_3 = t$. The solutions of the system are

$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ s \\ t \\ 0 \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ s, t \in \mathbb{R} \quad \text{or equivalently } \vec{\mathbf{x}} \in \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}$$

The solution space has dimension 2 with basis $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}$.

5. [2360/101922 (22 pts)] The following parts are not related.

(a) (10 pts) The characteristic polynomial of a certain matrix $\mathbf{B}$ is $p(\lambda) = 2\lambda^6 + 6\lambda^5 + 5\lambda^4$.

i. (2 pts) What is the order/size of the matrix $\mathbf{B}$?

ii. (8 pts) Find the eigenvalues of $\mathbf{B}$ and their (algebraic) multiplicities. Do not find the eigenvectors.

(b) (12 pts) Let $\mathbf{A} = \begin{bmatrix} -7 & 4 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 4 & -7 & -1 \\ 0 & 0 & 0 & -7 \end{bmatrix}$ and let $\mathbb{E}_{-7}$ denote the eigenspace associated with eigenvalue $\lambda = -7$. Find a basis for and the dimension of $\mathbb{E}_{-7}$.

SOLUTION:

(a) i. Since the characteristic polynomial is degree 6, $\mathbf{B}$ is of order 6 or 6×6.

ii.

$$2\lambda^6 + 6\lambda^5 + 5\lambda^4 = 0$$

$$\lambda^4(2\lambda^2 + 6\lambda + 5) = 0 \implies \lambda = 0 \quad \text{or} \quad 2\lambda^2 + 6\lambda + 5 = 0$$

$$\lambda = \frac{-6 \pm \sqrt{6^2 - 4(2)(5)}}{(2)(2)} = \frac{-6 \pm \sqrt{-4}}{4} = \frac{-6 \pm 2i}{4} = \frac{-3 \pm i}{2}$$

The eigenvalues of $\mathbf{B}$ are 0 with multiplicity of 4 and $\frac{-3+i}{2}, \frac{-3-i}{2}$, each with multiplicity of 1.

(b) We need to solve $(\mathbf{A} + 7\mathbf{I})\vec{\mathbf{v}} = \vec{\mathbf{0}}$

$$\left[\begin{array}{cccc|c} 0 & 4 & 0 & -1 & 0 \\ 0 & 8 & 0 & -2 & 0 \\ 0 & 4 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_2^* = -2R_1 + R_2 \\ R_3^* = -1R_1 + R_3 \\ R_1^* = \frac{1}{4}R_1 \end{array} \overset{\text{RREF}}{\longrightarrow} \left[\begin{array}{cccc|c} 0 & 1 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \implies \begin{array}{l} v_1 = r \\ v_2 = \frac{1}{4}v_4 = \frac{1}{4}t \\ v_3 = s \\ v_4 = t \end{array}$$

A basis for $\mathbb{E}_{-7}$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 4 \end{bmatrix} \right\}$ which has dimension 3.

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