1. [2360/092122 (35 pts)] Consider the initial value problem $(t+1)y' - 3(t+1)y + e^{3t} = 0,\ y(0) = \ln 3,\ t > -1$.

   (a) (4 pts) Classify the equation.

   (b) (2 pts) Does the equation possess any equilibrium solutions? If so, find them.

   (c) (7 pts) Is a unique solution guaranteed by Picard's Theorem? Justify your answer.

   (d) (7 pts) Use one step of Euler's Method to approximate y when $t = 1/10$. Simplify your answer.

   (e) (15 pts) Suppose the equation describes the amount of water in a well (in millions of gallons) with t the time in years. Will the well run dry? If so, when. If not, explain why not.

SOLUTION:

Begin by rewriting the equation as $y' = 3y - \dfrac{e^{3t}}{t+1}$

(a) first order, linear, constant coefficient, nonhomogeneous

(b) No. There are no constant values of y that will make the right hand side vanish.

(c) $f_y(t,y) = 3$ is continuous everywhere. For $t \neq -1$, $f(t,y) = 3y - \dfrac{e^{3t}}{t+1}$ consists of differences and quotients of continuous functions and is thus continuous in a rectangle surrounding $(0, \ln 3)$. Picard's Theorem guarantees the existence of a unique solution to the initial value problem.

(d)

$$y(0.1) \approx y_1 = y_0 + hf(t_0, y_0) = \ln 3 + 0.1\left(3\ln 3 - \frac{e^0}{0+1}\right) = 1.3\ln 3 - 0.1$$

(e) We need to solve the DE, which will be expedited by rewriting it as $y' - 3y = -\dfrac{e^{3t}}{t+1}$. Only one solution method is required.

Method 1 - Integrating factor

The integrating factor is $\mu(t) = e^{-3t}$ so that

$$\left(e^{-3t}y\right)' = -\frac{1}{t+1}$$

$$e^{-3t}y = -\ln|t+1| + C \qquad \text{(apply initial condition)}$$

$$e^0 \ln 3 = -\ln|0+1| + C \implies C = \ln 3 \qquad \text{(absolute value not needed since } t+1>0 \text{ on given interval)}$$

$$y = e^{3t}\ln\left(\frac{3}{t+1}\right)$$

Method 2 - Euler-Lagrange Two Stage/Variation of Parameters

Solve the associated homogeneous problem $\left(\dfrac{\mathrm{d}y_h}{\mathrm{d}t} - 3y_h = 0\right)$ using separation of variables.

$$\int \frac{\mathrm{d}y_h}{y_h} = \int 3\mathrm{d}t$$

$$\ln|y_h| = 3t + k$$

$$y_h(t) = Ce^{3t}$$

Set $y_p = v(t)y_h(t) = v(t)e^{3t}$ and substitute into the original nonhomogeneous equation.

$$y_p' - 3y_p = 3v(t)e^{3t} + v'(t)e^{3t} - 3v(t)e^{3t} = -\frac{e^{3t}}{t+1}$$

$$\int v'(t)\,\mathrm{d}t = \int -\frac{1}{t+1}\,\mathrm{d}t$$

$$v(t) = -\ln|t+1|$$

$$\implies y_p(t) = -e^{3t}\ln(t+1)$$

(absolute value not needed since $t + 1 > 0$ on given interval). By the Nonhomogeneous Principle the general solution is $y(t) = y_h(t) + y_p(t) = Ce^{3t} - e^{3t}\ln(t + 1)$ to which we apply the initial condition, giving $\ln 3 = C$ and the solution to the initial value problem as $y(t) = e^{3t}[\ln 3 - \ln(t + 1)] = e^{3t}\ln\left(\dfrac{3}{t+1}\right)$.

The well will go dry if $y = 0$ which occurs when $\dfrac{3}{t+1} = 1$ or $t = 2$ years.

2. [2360/092122 (15 pts)] Consider the differential equation $y' - y^2 + y^3 = 0$.

(a) (2 pts) Classify the equation.

(b) (4 pts) Find the equilibrium solutions.

(c) (5 pts) Plot the phase line.

(d) (2 pts) Determine the stability of all equilibrium solutions.

(e) (2 pts) Find the solution that passes through $(1, 1)$. Hint: Very little work is required to answer this.

SOLUTION:

(a) first order, nonlinear (autonomous as well but not required)

(b) $y' = y^2 - y^3 = y^2(1 - y) \implies y = 0$ and $y = 1$ are equilibrium solutions.

(c) $y < 0 \implies y' > 0; \quad 0 < y < 1 \implies y' > 0; \quad y > 1 \implies y' < 0$

(d) $y = 0$ is semistable and $y = 1$ is stable.

(e) $y = 1$ (the equilibrium solution)

3. [2360/092122 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

(a) Solutions to the differential equation $y' = -x^2 - y^4 - 3$ are always decreasing.

(b) The operator $L(\vec{\mathbf{y}}) = 2y + (1 - y)y^{(5)}$ is a linear operator

(c) The isocline of the differential equation $x' - e^t + 1 = 0$ corresponding to a slope of -2 does not exist.

(d) The substitution $u = y - 2x + 3$ makes the differential equation $\dfrac{dy}{dx} = 2 + \sqrt{y - 2x + 3}$ separable.

(e) The system of differential equations below has a single equilibrium solution at $(0, 0)$.

$$x' = 2 - x^2 - y^2$$
$$y' = y^2 - x$$

SOLUTION:

(a) **TRUE** The slope of the solution is negative for all values of x and y.

(b) **FALSE** y and its fifth derivative are multiplied together.

(c) **TRUE** For a given slope c the isoclines are $e^t - 1 = c$. If $c = -2$ this becomes $e^t = -1$ which has no solution.

(d) **TRUE** With $u = y - 2x + 3$ we have $\dfrac{du}{dx} = \dfrac{dy}{dx} - 2$ and the equation becomes

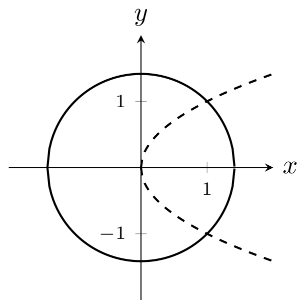
$$\frac{du}{dx} + 2 = 2 + \sqrt{u}$$
$$\frac{du}{\sqrt{u}} = dx$$

(e) **FALSE** Substituting $x = 0$ and $y = 0$ into the first equation gives $x' = 2 \neq 0$. Going a bit further, the v-nullcline is $x^2 + y^2 = 2$ and the h-nullcline is $y^2 = x$. Equilibrium points occur at the intersection of the h- and v-nullclines, so we seek solutions to the nonlinear system of equations

$$x^2 + y^2 = 2$$
$$y^2 = x$$

Substituting the second into the first yields $x^2 + x - 2 = 0 \implies (x+2)(x-1) = 0 \implies x = -2, 1$. Using these values in the second equation we have $y^2 = -2$ which has no solution and $y^2 = 1$ which has solutions $y = \pm 1$. The equilibrium solutions are thus $(1, 1)$ and $(1, -1)$.

Graphically, the dashed curve is the h-nullcline and the solid curve the v-nullcline. Their intersection points are the equilibrium solutions.



4. [2360/092122 (15 pts)] A 1000 gallon pot is initially 80 percent full of sweet tea in which 100 ounces of sugar is dissolved. Tea containing $1/(t + 1)$ ounces of sugar per gallon enters the pot at 5 gallons per minute. The well-mixed sweet tea leaves the pot at 7 gallons per minute.

   (a) (12 pts) Set up, but **DO NOT SOLVE**, an initial value problem for the amount of sugar, S, contained in the pot at time t.

   (b) (3 pts) If the initial time is $t = 0$, over what interval will the solution be valid? You do not need to find the solution to answer this question.

SOLUTION:

(a) Since the flow rates differ, the volume of sweet tea in the pot will vary with time. To determine this,

$$\frac{dV}{dt} = \text{flow rate in} - \text{flow rate out} = 5 - 7 = -2,\ V(0) = 800$$
$$\int dV = \int -2\, dt$$
$$V(t) = -2t + C$$
$$V(0) = 800 = 2(0) + C$$
$$V(t) = -2t + 800$$

$$\frac{dS}{dt} = \text{mass rate in} - \text{mass rate out} = \left(\frac{1}{t+1}\frac{\text{ounce}}{\text{gallon}}\right)\left(5\,\frac{\text{gallon}}{\text{minute}}\right) - \left(\frac{S}{-2t+800}\frac{\text{ounce}}{\text{gallon}}\right)\left(7\,\frac{\text{gallon}}{\text{minute}}\right)$$
$$\frac{dS}{dt} + \frac{7S}{800 - 2t} = \frac{5}{t+1},\ S(0) = 100$$

(b) The tank will be empty when $t = 400$, so the interval over which the solution of the differential equation is valid is $[0, 400]$.

5. [2360/092122 (25 pts)] Consider the initial value problem $x\frac{dw}{dx} + (2x + 1)w = 2x^2,\ w(1) = \frac{3}{2},\ x > 0$.

   (a) (5 pts) Without solving the differential equation, show that $w_p(x) = \frac{1}{2x} + x - 1$ is a particular solution.

(b) (15 pts) Find the general solution to the differential equation.

(c) (5 pts) Solve the initial value problem.

SOLUTION:

(a) Substitute w_p into the differential equation and show that an identity results.

$$x\frac{\mathrm{d}w_p}{\mathrm{d}x} + (2x+1)w_p \stackrel{?}{=} 2x^2$$

$$x\left(-\frac{1}{2x^2}+1\right) + (2x+1)\left(\frac{1}{2x}+x-1\right) \stackrel{?}{=} 2x^2$$

$$-\frac{1}{2x} + x + 1 + 2x^2 - 2x + \frac{1}{2x} + x - 1 \stackrel{?}{=} 2x^2$$

$$2x^2 = 2x^2 \quad \checkmark$$

(b) We need the solution, w_h, to the associated homogeneous equation.

$$x\frac{\mathrm{d}w_h}{\mathrm{d}x} + (2x+1)w_h = 0$$

$$\int \frac{\mathrm{d}w_h}{w_h} = \int -\frac{2x+1}{x}\,\mathrm{d}x = \int \left(-2-\frac{1}{x}\right)\mathrm{d}x$$

$$\ln|w_h| = -2x - \ln|x| + c = -2x - \ln x + c \quad \text{since } x > 0$$

$$|w_h| = e^{-2x-\ln x+c}$$

$$w_h = \frac{C}{xe^{2x}}, \quad C \in \mathbb{R}$$

Now apply the Nonhomogeneous Principle to obtain the general solution as

$$w = w_h + w_p = \frac{C}{xe^{2x}} + \frac{1}{2x} + x - 1$$

(c) Apply the initial condition.

$$w(1) = \frac{C}{e^2} + \frac{1}{2} + 1 - 1 = \frac{3}{2} \implies C = e^2$$

giving the solution to the initial value problem as

$$w(x) = \frac{e^{2-2x}}{x} + \frac{1}{2x} + x - 1$$

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