Consider the initial value problem \((t + 1)y' - 3(t + 1)y + e^{3t} = 0, \ y(0) = \ln 3, \ t > -1\).

(a) \(4\) pts Classify the equation.

(b) \(2\) pts Does the equation possess any equilibrium solutions? If so, find them.

(c) \(7\) pts Is a unique solution guaranteed by Picard’s Theorem? Justify your answer.

(d) \(7\) pts Use one step of Euler’s Method to approximate \(y\) when \(t = 1/10\). Simplify your answer.

(e) \(15\) pts Suppose the equation describes the amount of water in a well (in millions of gallons) with \(t\) the time in years. Will the well run dry? If so, when. If not, explain why not.

**SOLUTION:**

Begin by rewriting the equation as \(y' = 3y - \frac{e^{3t}}{t + 1}\)

(a) first order, linear, constant coefficient, nonhomogeneous

(b) No. There are no constant values of \(y\) that will make the right hand side vanish.

(c) \(f_y(t, y) = 3\) is continuous everywhere. For \(t \neq -1\), \(f(t, y) = 3y - \frac{e^{3t}}{t + 1}\) consists of differences and quotients of continuous functions and is thus continuous in a rectangle surrounding \((0, \ln 3)\). Picard’s Theorem guarantees the existence of a unique solution to the initial value problem.

(d)

\[
y(0.1) \approx y_1 = y_0 + hf(t_0, y_0) = \ln 3 + 0.1 \left(3\ln 3 - \frac{e^0}{0+1}\right) = 1.3\ln 3 - 0.1
\]

(e) We need to solve the DE, which will be expedited by rewriting it as \(y' - 3y = -\frac{e^{3t}}{t + 1}\). Only one solution method is required.

**Method 1 - Integrating factor**

The integrating factor is \(\mu(t) = e^{-3t}\) so that

\[
(e^{-3t}y)' = -\frac{1}{t + 1}
\]

\[
e^{-3t}y = -\ln|t + 1| + C \quad \text{(apply initial condition)}
\]

\[
e^0\ln 3 = -\ln|0 + 1| + C \quad \implies C = \ln 3 \quad \text{(absolute value not needed since } t + 1 > 0 \text{ on given interval)}
\]

\[
y = e^{3t}\ln\left(\frac{3}{t + 1}\right)
\]

**Method 2 - Euler-Lagrange Two Stage/Variation of Parameters**

Solve the associated homogeneous problem \(\left(\frac{dy_h}{dt} - 3y_h = 0\right)\) using separation of variables.

\[
\int \frac{dy_h}{y_h} = \int 3\,dt
\]

\[
\ln|y_h| = 3t + k
\]

\[
y_h(t) = Ce^{3t}
\]

Set \(y_p = v(t)y_h(t) = v(t)e^{3t}\) and substitute into the original nonhomogeneous equation.

\[
y_p' - 3y_p = 3v(t)e^{3t} + v'(t)e^{3t} - 3v(t)e^{3t} = -\frac{e^{3t}}{t + 1}
\]

\[
\int v'(t)\,dt = \int -\frac{1}{t + 1}\,dt
\]

\[
v(t) = -\ln|t + 1|
\]

\[
\implies y_p(t) = -e^{3t}\ln(t + 1)
\]
(absolute value not needed since $t + 1 > 0$ on given interval). By the Nonhomogeneous Principle the general solution is $y(t) = y_h(t) + y_p(t) = Ce^{3t} - e^{3t} \ln(t + 1)$ which we apply the initial condition, giving $\ln 3 = C$ and the solution to the initial value problem as $y(t) = e^{3t} [\ln 3 - \ln(t + 1)] = e^{3t} \ln \left( \frac{3}{t + 1} \right)$. 

The well will go dry if $y = 0$ which occurs when $\frac{3}{t + 1} = 1$ or $t = 2$ years.

2. Consider the differential equation $y' - y^2 + y^3 = 0$.

(a) Classify the equation.
(b) Find the equilibrium solutions.
(c) Plot the phase line.
(d) Determine the stability of all equilibrium solutions.
(e) Find the solution that passes through $(1, 1)$. Hint: Very little work is required to answer this.

**SOLUTION:**

(a) first order, nonlinear (autonomous as well but not required)
(b) $y' = y^2 - y^3 = y^2(1 - y) \implies y = 0$ and $y = 1$ are equilibrium solutions.
(c) $y < 0 \implies y' > 0$; $0 < y < 1 \implies y' > 0$; $y > 1 \implies y' < 0$

\[ \begin{array}{c}
\downarrow \\
1 \bullet \\
\uparrow \\
0 \bigcirc \\
\uparrow
\end{array} \]

(d) $y = 0$ is semistable and $y = 1$ is stable.
(e) $y = 1$ (the equilibrium solution)

3. Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

(a) Solutions to the differential equation $y' = -x^2 - y^4 - 3$ are always decreasing.
(b) The operator $L(y') = 2y + (1 - y)y^{(5)}$ is a linear operator
(c) The isocline of the differential equation $x' - e^t + 1 = 0$ corresponding to a slope of $-2$ does not exist.
(d) The substitution $u = y - 2x + 3$ makes the differential equation $\frac{dy}{dx} = 2 + \sqrt{y - 2x + 3}$ separable.
(e) The system of differential equations below has a single equilibrium solution at $(0, 0)$.

\[ \begin{align*}
x' &= 2 - x^2 - y^2 \\
y' &= y^2 - x
\end{align*} \]

**SOLUTION:**

(a) **TRUE** The slope of the solution is negative for all values of $x$ and $y$.
(b) **FALSE** $y$ and its fifth derivative are multiplied together.
(c) **TRUE** For a given slope $c$ the isoclines are $e^c - 1 = c$. If $c = -2$ this becomes $e^c = -1$ which has no solution.
(d) **TRUE** With $u = y - 2x + 3$ we have $\frac{du}{dx} = \frac{dy}{dx} - 2$ and the equation becomes

\[ \frac{du}{dx} + 2 = 2 + \sqrt{u} \]

\[ \frac{du}{\sqrt{u}} = dx \]
(e) **FALSE** Substituting $x = 0$ and $y = 0$ into the first equation gives $x' = 2 \neq 0$. Going a bit further, the $v$-nullcline is $x^2 + y^2 = 2$ and the $h$-nullcline is $y^2 = x$. Equilibrium points occur at the intersection of the $h$- and $v$-nullclines, so we seek solutions to the nonlinear system of equations

$$
x^2 + y^2 = 2
$$
$$
y^2 = x
$$

Substituting the second into the first yields $x^2 + x - 2 = 0 \implies (x + 2)(x - 1) = 0 \implies x = -2, 1$. Using these values in the second equation we have $y^2 = -2$ which has no solution and $y^2 = 1$ which has solutions $y = \pm 1$. The equilibrium solutions are thus $(1, 1)$ and $(1, -1)$.

Graphically, the dashed curve is the $h$-nullcline and the solid curve the $v$-nullcline. Their intersection points are the equilibrium solutions.

![Graph](image)

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4. [2360/092122 (15 pts)] A 1000 gallon pot is initially 80 percent full of sweet tea in which 100 ounces of sugar is dissolved. Tea containing $\frac{1}{t+1}$ ounces of sugar per gallon enters the pot at 5 gallons per minute. The well-mixed sweet tea leaves the pot at 7 gallons per minute.

(a) (12 pts) Set up, but **DO NOT SOLVE**, an initial value problem for the amount of sugar, $S$, contained in the pot at time $t$.

(b) (3 pts) If the initial time is $t = 0$, over what interval will the solution be valid? You do not need to find the solution to answer this question.

**Solution:**

(a) Since the flow rates differ, the volume of sweet tea in the pot will vary with time. To determine this,

$$
\frac{dV}{dt} = \text{flow rate in} - \text{flow rate out} = 5 - 7 = -2, \quad V(0) = 800
$$

\[\int dV = \int -2 \, dt \]

\[V(t) = -2t + C\]

\[V(0) = 800 = 2(0) + C\]

\[V(t) = -2t + 800\]

$$
\frac{dS}{dt} = \text{mass rate in} - \text{mass rate out} = \left(\frac{1}{t+1} \text{ ounce} \frac{1}{\text{gallon}}\right) \left(5 \text{ gallon} \frac{1}{\text{minute}}\right) - \left(\frac{S}{-2t + 800} \text{ ounce} \frac{1}{\text{gallon}}\right) \left(7 \text{ gallon} \frac{1}{\text{minute}}\right)
$$

\[\frac{dS}{dt} + \frac{7S}{800 - 2t} = \frac{5}{t + 1}, \quad S(0) = 100\]

(b) The tank will be empty when $t = 400$, so the interval over which the solution of the differential equation is valid is $[0, 400]$.

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5. [2360/092122 (25 pts)] Consider the initial value problem $x \frac{dw}{dx} + (2x + 1)w = 2x^2, \quad w(1) = \frac{3}{2}, \quad x > 0$.

(a) (5 pts) Without solving the differential equation, show that $w_p(x) = \frac{1}{2x} + x - 1$ is a particular solution.
(b) (15 pts) Find the general solution to the differential equation.

(c) (5 pts) Solve the initial value problem.

**SOLUTION:**

(a) Substitute $w_p$ into the differential equation and show that an identity results.

$$x \frac{dw_p}{dx} + (2x + 1)w_p = 2x^2$$

$$x \left( -\frac{1}{2x^2} + 1 \right) + (2x + 1) \left( \frac{1}{2x} + x - 1 \right) = 2x^2$$

$$-\frac{1}{2x} + x + 1 + 2x^2 - 2x + \frac{1}{2x} + x - 1 = 2x^2$$

$$2x^2 = 2x^2 \quad \checkmark$$

(b) We need the solution, $w_h$, to the associated homogeneous equation.

$$x \frac{dw_h}{dx} + (2x + 1)w_h = 0$$

$$\int \frac{dw_h}{w_h} = \int -\frac{2x + 1}{x} \, dx = \int \left( -2 - \frac{1}{x} \right) \, dx$$

$$\ln |w_h| = -2x - \ln |x| + c = -2x - \ln x + c \quad \text{since } x > 0$$

$$|w_h| = e^{-2x - \ln x + c}$$

$$w_h = \frac{C}{xe^{2x}}, \quad C \in \mathbb{R}$$

Now apply the Nonhomogeneous Principle to obtain the general solution as

$$w = w_h + w_p = \frac{C}{xe^{2x}} + \frac{1}{2x} + x - 1$$

(c) Apply the initial condition.

$$w(1) = \frac{C}{e^2} + \frac{1}{2} + 1 - 1 = \frac{3}{2} \implies C = e^2$$

giving the solution to the initial value problem as

$$w(x) = \frac{e^{2-2x}}{x} + \frac{1}{2x} + x - 1$$