1. [2360/121121 (14 pts)] In your bluebook, in a column, write the letters (a)-(g) and next to each letter write the word **TRUE** or **FALSE** as appropriate. You need not show any work and no partial credit will be given.

   (a) $y = t\ln t$ is a solution to the initial value problem $(y'')^2 - 3ty' + 3y = \dfrac{1-3t^3}{t^2}$, $y(e) = e$, $y'(e) = 2$ on the interval $t > 0$.

   (b) The set $\mathbb{W}$ of all 2×2 singular matrices is a subspace of $\mathbb{M}_{22}$.

   (c) The oscillator governed by the differential equation $2\ddot{x} + 98x = -7\cos 7t$ is in resonance.

   (d) If $\mathbf{AC}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a unique solution for all $\vec{\mathbf{b}}$, then $|\mathbf{A}| = 0$.

   (e) $2t^2 + t$ is in span $\{1, 1-t, t^2\}$.

   (f) $(\sin^2 x + 1)\, y = \sqrt{x}\, y'$ is a separable, linear, nonhomogeneous differential equation.

   (g) If $\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & -2 \end{bmatrix}$, then $\mathbf{AA}^{\mathrm{T}} = \begin{bmatrix} 6 & -5 \\ -5 & 13 \end{bmatrix}$.

   SOLUTION:

   (a) **TRUE** $y' = 1 + \ln t, y'' = \dfrac{1}{t}, y(e) = e, y'(e) = 2$ and

   $$(y'')^2 - 3ty' + 3y = \left(\frac{1}{t}\right)^2 - 3t(1+\ln t) + 3(t\ln t) = \left(\frac{1}{t}\right)^2 - 3t = \frac{1-3t^3}{t^2}$$

   (b) **FALSE** $\mathbb{W}$ is not closed under addition. For example,

   $$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \in \mathbb{W} \text{ and } \begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix} \in \mathbb{W} \text{ but } \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \notin \mathbb{W}$$

   (c) **TRUE** $\omega_0 = \sqrt{\dfrac{98}{2}} = \sqrt{49} = 7 = \omega_f$

   (d) **FALSE** If the system has a unique solution, then $\mathbf{AC}$ is invertible. Since $(\mathbf{AC})^{-1} = \mathbf{C}^{-1}\mathbf{A}^{-1}$, this implies $\mathbf{A}$ is invertible, further implying that $|\mathbf{A}| \neq 0$.

   (e) **TRUE** $(1)(1) + (-1)(1-t) + (2)(t^2) = 2t^2 + t$

   (f) **FALSE** The equation is linear (only y and y' present) , separable $\left(\dfrac{\mathrm{d}y}{y} = \dfrac{\sin^2 x + 1}{\sqrt{x}}\mathrm{d}x\right)$, and homogeneous $\sqrt{x}y' + (\sin^2 x + 1)\, y = 0$

   (g) **TRUE** $\mathbf{AA}^{\mathrm{T}} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 6 & -5 \\ -5 & 13 \end{bmatrix}$

   ■

2. [2360/121121 (20 pts)] Let $\mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}$.

   (a) (10 pts) Find the eigenvalues of $\mathbf{B}$. Hint: $x^3 - x^2 + x - 1 = (x-1)(x^2+1)$.

   (b) (10 pts) Find a basis for the eigenspace associated with the real eigenvalue and determine its dimension.

   SOLUTION:

   (a) Expanding across the first row gives

   $$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ -1 & 1 & -\lambda \end{vmatrix} = (1-\lambda)(-1)^{1+1}\begin{vmatrix} -\lambda & 0 \\ 1 & -\lambda \end{vmatrix} + (1)(-1)^{1+3}\begin{vmatrix} 1 & -\lambda \\ -1 & 1 \end{vmatrix}$$
   $$= (1-\lambda)\lambda^2 + 1 - \lambda = -\lambda^3 + \lambda^2 - \lambda + 1$$
   $$= -\left(\lambda^3 - \lambda^2 + \lambda - 1\right) = -(\lambda - 1)\left(\lambda^2 + 1\right) = 0$$

   giving $\lambda = 1, \pm i$ as the eigenvalues.

(b) To find the eigenvector associated with $\lambda = 1$, we need to find nontrivial solutions to $(\mathbf{B} - \mathbf{I})\,\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

$$\left[\begin{array}{rrr|r} 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{rrr|r} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \vec{\mathbf{v}} = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

so a basis for the eigenspace is $\left\{ \begin{bmatrix}1\\1\\0\end{bmatrix} \right\}$ with dimension 1.

3. [2360/121121 (21 pts)] Consider the matrix $\mathbf{A} = \begin{bmatrix}-3 & 2\\ -1 & -1\end{bmatrix}$ that has $-2+i$ as one of its eigenvalues.

(a) (2 pts) What is the other eigenvalue?

(b) (9 pts) Let $\vec{\mathbf{v}} = \begin{bmatrix}1-i\\ 1\end{bmatrix}$.

i. (3 pts) Compute $\mathbf{A}\vec{\mathbf{v}}$.

ii. (3 pts) Compute $(-2+i)\vec{\mathbf{v}}$.

iii. (3 pts) What do the two previous calculations allow you to conclude about $\vec{\mathbf{v}}$?

(c) (10 pts) Solve $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ if $\vec{\mathbf{x}}(0) = \begin{bmatrix}-1\\ 1\end{bmatrix}$, writing your answer as a single vector. Hint: Most of the work necessary to solve this has been done already.

SOLUTION:

(a) Since the entries of $\mathbf{A}$ are real, eigenvalues come in complex conjugate pairs so the other eigenvalue is $-2-i$.

(b) i.

$$\mathbf{A}\vec{\mathbf{v}} = \begin{bmatrix}-3 & 2\\ -1 & -1\end{bmatrix}\begin{bmatrix}1-i\\ 1\end{bmatrix} = \begin{bmatrix}-3+3i+2\\ -1+i-1\end{bmatrix} = \begin{bmatrix}-1+3i\\ -2+i\end{bmatrix}$$

ii.

$$(-2+i)\vec{\mathbf{v}} = (-2+i)\begin{bmatrix}1-i\\ 1\end{bmatrix} = \begin{bmatrix}-2+2i+i+1\\ -2+i\end{bmatrix} = \begin{bmatrix}-1+3i\\ -2+i\end{bmatrix}$$

iii. Since $\mathbf{A}\vec{\mathbf{v}} = (2+i)\vec{\mathbf{v}}$, $\vec{\mathbf{v}}$ is the eigenvector associated with eigenvalue $-2+i$.

(c) Eigenvalue $-2+i$ gives $\alpha = -2$ and $\beta = 1$. Eigenvector $\vec{\mathbf{v}} = \begin{bmatrix}1-i\\ 1\end{bmatrix} = \begin{bmatrix}1\\ 1\end{bmatrix} + i\begin{bmatrix}-1\\ 0\end{bmatrix}$ gives $\vec{\mathbf{p}} = \begin{bmatrix}1\\ 1\end{bmatrix}$ and $\vec{\mathbf{q}} = \begin{bmatrix}-1\\ 0\end{bmatrix}$. This gives the general solution as

$$\vec{\mathbf{x}}(t) = c_1 e^{-2t}\left(\cos t\begin{bmatrix}1\\ 1\end{bmatrix} - \sin t\begin{bmatrix}-1\\ 0\end{bmatrix}\right) + c_2 e^{-2t}\left(\sin t\begin{bmatrix}1\\ 1\end{bmatrix} + \cos t\begin{bmatrix}-1\\ 0\end{bmatrix}\right)$$

Applying the initial condition results in

$$c_1\begin{bmatrix}1\\ 1\end{bmatrix} + c_2\begin{bmatrix}-1\\ 0\end{bmatrix} = \begin{bmatrix}-1\\ 1\end{bmatrix} \implies \begin{array}{c} c_1 - c_2 = -1 \\ c_1 = 1 \end{array} \implies c_1 = 1,\ c_2 = 2$$

giving the solution to the initial value problem as $\vec{\mathbf{x}}(t) = e^{-2t}\begin{bmatrix}3\sin t - \cos t\\ 2\sin t + \cos t\end{bmatrix}$

4. [2360/121121 (22 pts)] The rate of change of the temperature, $T(t)$, of a certain object is equal to $-2\left[T - (1 + e^{-t})\right]$ where t is the time. The initial $(t = 0)$ temperature of the object is 6.

(a) (5 pts) Does Picard's theorem guarantee the existence of a unique solution to this initial value problem? Justify your answer.

(b) (5 pts) With stepsize 0.1, write the equation of Euler's Method to approximate the differential equation's solution numerically. Use variables t and T in the equation.

(c) (10 pts) Solve the exact differential equation to find $T(t)$, the temperature at any time t.

(d) (2 pts) What is the steady state temperature, that is, what is the temperature after a "long" time?

SOLUTION:

(a) Here the dependent variable is T (corresponding to y) and the independent variable is t. We have $f(t,T) = -2\left[T - (1+e^{-t})\right]$ which is a sum of functions that are continuous for all values of t and T and thus itself is continuous for all values of t and T and therefore in a rectangle containing $(0,6)$. Furthermore, $f_T(t,T) = -2$ which is continuous everywhere and hence in a rectangle containing $(0,6)$. Picard's Theorem guarantees a unique solution to the initial value problem.

(b) $T_{n+1} = T_n + 0.1(-2)\left[(T_n - (1+e^{-t_n})\right]$

(c) We need to solve the initial value problem $\dfrac{\mathrm{d}T}{\mathrm{d}t} = -2T + 2 + 2e^{-t}$, $T(0) = 6$.

Euler-Lagrange Two Stage Method (variation of parameters):

$$\frac{\mathrm{d}T_h}{\mathrm{d}t} = -2T_h$$
$$\int \frac{\mathrm{d}T_h}{T_h} = -2\int \mathrm{d}t$$
$$\ln|T_h| = -2t + k$$
$$|T_h| = e^k e^{-2t}$$
$$T_h = ce^{-2t}$$

Let $T_p = v(t)e^{-2t}$ and substitute into the differential equation to get

$$-2v(t)e^{-2t} + v'(t)e^{-2t} + 2v(t)e^{-2t} = 2 + 2e^{-t}$$
$$v'(t) = 2e^{2t} + 2e^{t}$$
$$v(t) = e^{2t} + 2e^{t}$$
$$T_p = \left(e^{2t} + 2e^{t}\right)e^{-2t} = 1 + 2e^{-t}$$

Thus $T(t) = T_h + T_p = ce^{-2t} + 1 + 2e^{-t}$ and applying the initial condition gives $T(0) = c + 1 + 2 = 6 \implies c = 3$ so that

$$T(t) = 3e^{-2t} + 2e^{-t} + 1$$

Integrating Factor Method:

We can write the differential equation as $\dfrac{\mathrm{d}T}{\mathrm{d}t} + 2T = 2 + 2e^{-t}$ so that $p(t) = 2$ yielding an integrating factor of $\mu(t) = e^{2t}$. Then

$$e^{2t}\left(\frac{\mathrm{d}T}{\mathrm{d}t} + 2T = 2 + 2e^{-t}\right)$$
$$\left(e^{2t}T\right)' = 2e^{2t} + 2e^{t}$$
$$e^{2t}T = e^{2t} + 2e^{t} + c$$
$$T(t) = 1 + 2e^{-t} + ce^{-2t}$$

Applying the initial condition gives $T(0) = 1 + 2 + c = 6 \implies c = 3$ so that

$$T(t) = 3e^{-2t} + 2e^{-t} + 1$$

(d) $\lim\limits_{t\to\infty} T(t) = 1$ which is the steady state temperature.

■

5. [2360/121121 (20 pts)] Use the method of undetermined coefficients to solve the initial value problem $\dfrac{\mathrm{d}^3y}{\mathrm{d}t^3} - \dfrac{\mathrm{d}y}{\mathrm{d}t} = 12e^{2t}$ with initial conditions $y(0) = 2,\ y'(0) = 4,\ y''(0) = 11$. Use Cramer's rule to solve any linear system of algebraic equations that should arise.

SOLUTION:

Solve the associated homogeneous equation $y_h''' - y_h' = 0$ whose characteristic equation is $r^3 - r = r\left(r^2-1\right) = r(r+1)(r-1) = 0$ with roots $0, 1, -1$ giving $y_h = c_1 + c_2 e^t + c_3 e^{-t}$.
We guess that $y_p = Ae^{2t}$.

$$y_p''' - y_p' = 8Ae^{2t} - 2Ae^{2t} = 6Ae^{2t} = 12e^{2t} \implies A = 2 \implies y_p = 2e^{2t}$$

so that the general solution is

$$\begin{aligned} y(t) &= y_h + y_p = c_1 + c_2 e^t + c_3 e^{-t} + 2e^{2t} \qquad \text{with} \\ y'(t) &= c_2 e^t - c_3 e^{-t} + 4e^{2t} \qquad \text{and} \\ y''(t) &= c_2 e^t + c_3 e^{-t} + 8e^{2t} \end{aligned}$$

Applying the initial conditions yields

$$\begin{aligned} c_1 + c_2 + c_3 + 2 &= 2 \\ c_2 - c_3 + 4 &= 4 \\ c_2 + c_3 + 8 &= 11 \end{aligned}$$

which can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

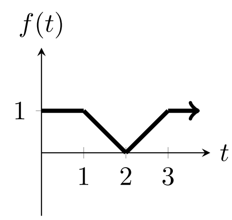
Using Cramer's Rule to solve we have

$$c_1 = \frac{\begin{vmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 3 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix}} = \frac{-6}{2} = -3 \qquad c_2 = \frac{\begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 3 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix}} = \frac{3}{2} \qquad c_3 = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix}} = \frac{3}{2}$$

The solution to the initial value problem is $y(t) = -3 + \frac{3}{2}e^t + \frac{3}{2}e^{-t} + 2e^{2t}$. $\left[\text{Just for fun, } y(t) = 3(\cosh t - 1) + 2e^{2t}\right]$. ■

6. [2360/121121 (35 pts)] The following problems are not related.

   (a) (10 pts) Using the graph below, write the function $f(t)$, defined on $[0, \infty)$, as a single function using step functions.



   (b) (10 pts) Find the Laplace Transform of $f(t) = 7t^2\,\text{step}(t-3) + e^{-t+5}\,\text{step}(t-5)$.

   (c) (15 pts) Solve the initial value problem $y'' + y' = \delta(t-2)$, $y(0) = 1$, $y'(0) = 0$.

SOLUTION:

   (a) Any of the following is acceptable, noting that the first term, 1, can be replaced with step(t) in all of the them.

$$\begin{aligned} f(t) &= 1 - \text{step}(t-1) + (2-t)\text{step}(t-1) - (2-t)\text{step}(t-2) + (t-2)\text{step}(t-2) - (t-2)\text{step}(t-3) + \text{step}(t-3) \\ &= 1 - \text{step}(t-1) + (2-t)\left[\text{step}(t-1) - \text{step}(t-2)\right] + (t-2)\left[\text{step}(t-2) - \text{step}(t-3)\right] + \text{step}(t-3) \\ &= 1 - \text{step}(t-1) + (2-t)\left[\text{step}(t-1) - \text{step}(t-2)\right] - (2-t)\left[\text{step}(t-2) - \text{step}(t-3)\right] + \text{step}(t-3) \\ &= 1 - \text{step}(t-1) + (2-t)\left[\text{step}(t-1) - 2\,\text{step}(t-2) + \text{step}(t-3)\right] + \text{step}(t-3) \end{aligned}$$

   Note: The first term can be written as step(t).

(b) We can rewrite $f(t) = 7t^2\,\text{step}(t-3) + e^{-(t-5)}\,\text{step}(t-5)$. Then using the delay theorems (both forms) we have

$$\begin{aligned}\mathscr{L}\{f(t)\} &= 7e^{-3s}\mathscr{L}\left\{(t+3)^2\right\} + \frac{e^{-5s}}{s+1}\\ &= 7e^{-3s}\mathscr{L}\left\{t^2+6t+9\right\} + \frac{e^{-5s}}{s+1}\\ &= 7e^{-3s}\left(\frac{2}{s^3}+\frac{6}{s^2}+\frac{9}{s}\right) + \frac{e^{-5s}}{s+1}\end{aligned}$$

(c) Taking Laplace Transforms of both sides yields:

$$s^2Y(s) - sy(0) - y'(0) + sY(s) - y(0) = e^{-2s}$$

$$\left(s^2+s\right)Y(s) = e^{-2s} + s + 1$$

$$Y(s) = \frac{e^{-2s}}{s(s+1)} + \frac{s+1}{s(s+1)} \qquad \text{(simplification and partial fractions)}$$

$$Y(s) = e^{-2s}\left(\frac{1}{s} - \frac{1}{s+1}\right) + \frac{1}{s}$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \left[1 - e^{-(t-2)}\right]\text{step}(t-2) + 1$$

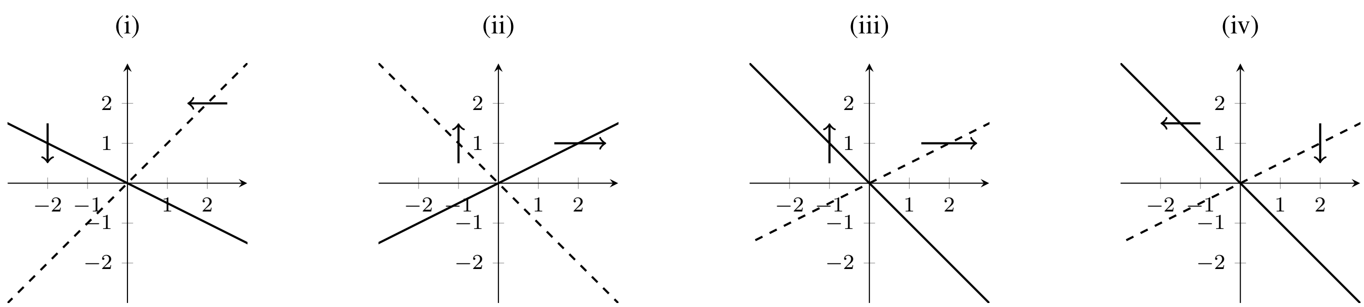
■

7. [2360/121121 (18 pts)] Consider the system of differential equations given by $\vec{\mathbf{x}}' = \begin{bmatrix} 1 & 1 \\ -2 & c \end{bmatrix}\vec{\mathbf{x}}$.

(a) (12 pts) Find all real values of c, if any, for which the isolated equilibrium solution (fixed point) at $(0,0)$ is

i. a center ii. asymptotically stable iii. a saddle iv. an unstable degenerate (improper) or star node

(b) (6 pts) Now let $c = 4$. The four graphs below show h nullclines (dashed), v nullclines (solid) and 2 elements of the vector field (arrows) associated with this system. In your bluebook, write down which graph correctly depicts these features of the given system of differential equations.



SOLUTION:

(a) Tr $\mathbf{A} = c+1$ and $|\mathbf{A}| = c+2$. Since we are only considering isolated equilibrium solutions, we must have $|\mathbf{A}| \neq 0$.

i. For a center, we need Tr $\mathbf{A} = 0$ and $|\mathbf{A}| > 0$. Thus $c = -1$.

ii. For asymptotic stability, we need to be in the second quadrant of the Tr $\mathbf{A}$-$|\mathbf{A}|$ diagram. Thus

$$\text{Tr}\,\mathbf{A} = c+1 < 0 \implies c < -1 \text{ and } |\mathbf{A}| = c+2 > 0 \implies c > -2 \text{ which combined require } -2 < c < -1$$

iii. For a saddle, we simply require $|\mathbf{A}| = c+2 < 0 \implies c < -2$

iv. For an unstable degenerate or star node we need $(\text{Tr}\,\mathbf{A})^2 - 4|\mathbf{A}| = 0, \text{Tr}\,\mathbf{A} > 0, |\mathbf{A}| > 0$. Thus

$$(c+1)^2 - 4(c+2) = 0$$

$$c^2 + 2c + 1 - 4c - 8 = 0$$

$$c^2 - 2c - 7 = 0$$

$$c = \frac{2 \pm \sqrt{32}}{2} = 1 \pm 2\sqrt{2}$$

Since we need Tr $\mathbf{A} > 0$ and Tr $\mathbf{A} = c + 1 = 2 \pm 2\sqrt{2}$ we have to choose the plus sign so that $c = 1 + 2\sqrt{2}$. Note that with this value of c, $|\mathbf{A}| = 3 + 2\sqrt{2} > 0$ which is also required.

(b) The system can be written as

$$x_1' = x_1 + x_2$$

$$x_2' = -2x_1 + 4x_2$$

The h nullcline ($x_2' = 0$) is $x_2 = \dfrac{1}{2}x_1$ and the v nullcline ($x_1' = 0$) is $x_2 = -x_1$. The point $(2, 1)$ is on the h nullcline ($x_2' = 0$) and $x_1' > 0$ there so the vector points to the right. The point $(-1, 1)$ is on the v nullcline ($x_1' = 0$) and $x_2' > 0$ there so the vector points upward. Graph (iii) is the correct one.

■