- This exam is worth 150 points and has 7 questions.
- **Show all work and simplify your answers!** Answers with no justification will receive no points unless otherwise noted.
- **Please begin each problem on a new page.**
- **DO NOT** leave the exam until you have satisfactorily scanned and uploaded your exam to Gradescope.
- You are taking this exam in a proctored and honor code enforced environment. **NO** calculators, cell phones, or other electronic devices or the internet are permitted. You are allowed one 8.5"× 11" crib sheet with writing on both sides.

0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/121121 (14 pts)] In your bluebook, in a column, write the letters (a)-(g) and next to each letter write the word **TRUE** or **FALSE** as appropriate. You need not show any work and no partial credit will be given.

   (a) $y = t\ln t$ is a solution to the initial value problem $(y'')^2 - 3ty' + 3y = \dfrac{1-3t^3}{t^2}$, $y(e) = e$, $y'(e) = 2$ on the interval $t > 0$.

   (b) The set $\mathbb{W}$ of all 2×2 singular matrices is a subspace of $\mathbb{M}_{22}$.

   (c) The oscillator governed by the differential equation $2\ddot{x} + 98x = -7\cos 7t$ is in resonance.

   (d) If $\mathbf{AC}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a unique solution for all $\vec{\mathbf{b}}$, then $|\mathbf{A}| = 0$.

   (e) $2t^2 + t$ is in span $\left\{1, 1-t, t^2\right\}$.

   (f) $\left(\sin^2 x + 1\right) y = \sqrt{x}\, y'$ is a separable, linear, nonhomogeneous differential equation.

   (g) If $\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & -2 \end{bmatrix}$, then $\mathbf{AA}^{\mathrm{T}} = \begin{bmatrix} 6 & -5 \\ -5 & 13 \end{bmatrix}$.

2. [2360/121121 (20 pts)] Let $\mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}$.

   (a) (10 pts) Find the eigenvalues of $\mathbf{B}$. Hint: $x^3 - x^2 + x - 1 = (x-1)(x^2+1)$.

   (b) (10 pts) Find a basis for the eigenspace associated with the real eigenvalue and determine its dimension.

3. [2360/121121 (21 pts)] Consider the matrix $\mathbf{A} = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$ that has $-2+i$ as one of its eigenvalues.

   (a) (2 pts) What is the other eigenvalue?

   (b) (9 pts) Let $\vec{\mathbf{v}} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$.

      i. (3 pts) Compute $\mathbf{A}\vec{\mathbf{v}}$.

      ii. (3 pts) Compute $(-2+i)\vec{\mathbf{v}}$.

      iii. (3 pts) What do the two previous calculations allow you to conclude about $\vec{\mathbf{v}}$?

   (c) (10 pts) Solve $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ if $\vec{\mathbf{x}}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, writing your answer as a single vector. Hint: Most of the work necessary to solve this has been done already.

4. [2360/121121 (22 pts)] The rate of change of the temperature, $T(t)$, of a certain object is equal to $-2\left[T - (1+e^{-t})\right]$ where t is the time. The initial ($t = 0$) temperature of the object is 6.

   (a) (5 pts) Does Picard's theorem guarantee the existence of a unique solution to this initial value problem? Justify your answer.

   (b) (5 pts) With stepsize 0.1, write the equation of Euler's Method to approximate the differential equation's solution numerically. Use variables t and T in the equation.

   (c) (10 pts) Solve the exact differential equation to find $T(t)$, the temperature at any time t.

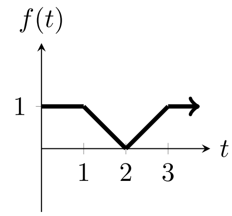
   (d) (2 pts) What is the steady state temperature, that is, what is the temperature after a "long" time?

PROBLEMS 5-7 AND LAPLACE TRANSFORM TABLE ON NEXT PAGE / OTHER SIDE

5. [2360/121121 (20 pts)] Use the method of undetermined coefficients to solve the initial value problem $\dfrac{\mathrm{d}^3y}{\mathrm{d}t^3} - \dfrac{\mathrm{d}y}{\mathrm{d}t} = 12e^{2t}$ with initial conditions $y(0) = 2$, $y'(0) = 4$, $y''(0) = 11$. Use Cramer's rule to solve any linear system of algebraic equations that should arise.

6. [2360/121121 (35 pts)] The following problems are not related.

   (a) (10 pts) Using the graph below, write the function $f(t)$, defined on $[0, \infty)$, as a single function using step functions.



   (b) (10 pts) Find the Laplace Transform of $f(t) = 7t^2\,\text{step}(t-3) + e^{-t+5}\,\text{step}(t-5)$.

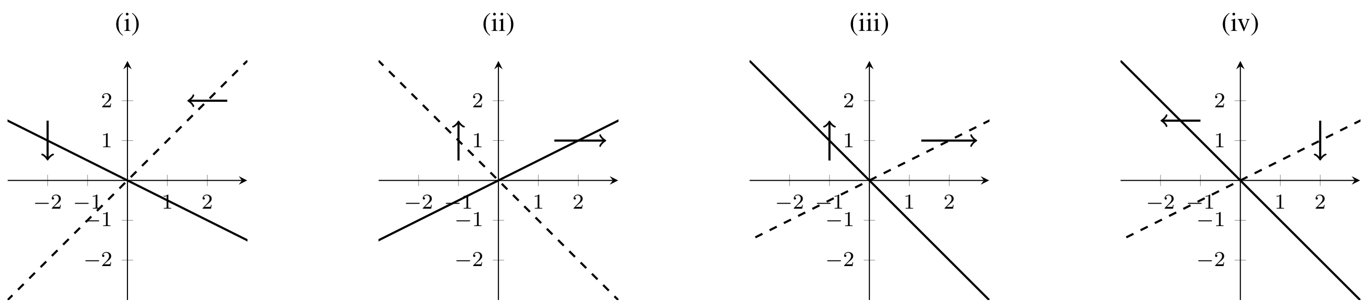
   (c) (15 pts) Solve the initial value problem $y'' + y' = \delta(t-2)$, $y(0) = 1$, $y'(0) = 0$.

7. [2360/121121 (18 pts)] Consider the system of differential equations given by $\vec{\mathbf{x}}' = \begin{bmatrix} 1 & 1 \\ -2 & c \end{bmatrix} \vec{\mathbf{x}}$.

   (a) (12 pts) Find all real values of c, if any, for which the isolated equilibrium solution (fixed point) at $(0, 0)$ is

   i. a center   ii. asymptotically stable   iii. a saddle   iv. an unstable degenerate (improper) or star node

   (b) (6 pts) Now let $c = 4$. The four graphs below show h nullclines (dashed), v nullclines (solid) and 2 elements of the vector field (arrows) associated with this system. In your bluebook, write down which graph correctly depicts these features of the given system of differential equations.



Short table of Laplace Transforms: $\mathscr{L}\{f(t)\} = F(s) \equiv \displaystyle\int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a, b, c are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{t f'(t)\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\{f(t-c)\,\text{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\text{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$
