1. [2360/111721 (20 pts)] Consider the differential equation $y'' - \dfrac{2}{t^2}y = -6$ for $t \geq 1$.

   (a) (10 pts) Show that the functions $y_1 = t^2$ and $y_2 = \dfrac{1}{t}$ form a basis for the solution space of the associated homogeneous problem. Justify your answer completely.

   (b) (10 pts) Find the general solution to the original nonhomogeneous problem.

SOLUTION:

(a) Begin by showing that y_1 and y_2 are solutions.

$$y_1 - \frac{2}{t^2}y_1 = \left(t^2\right)'' - \frac{2}{t^2}t^2 = 2 - 2 = 0$$
$$y_2 - \frac{2}{t^2}y_2 = \left(t^{-1}\right)'' - \frac{2}{t^2}t^{-1} = 2t^{-3} - 2t^{-3} = 0$$

Now show that y_1 and y_2 are linearly independent.

$$W[y_1, y_2](t) = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} = -3 \neq 0$$

so the functions are linearly independent on $t \geq 1$. Since the dimension of the solution space is 2 (second order equation) and we have two linearly independent solutions, they form a basis for the solution space of the association homogeneous equation.

(b) Since this is a variable coefficient differential equation, we must use variation of parameters. In this case, $y_p = v_1 y_1 + v_2 y_2$ and with $f(t) = -6$ we have

$$v_1' = -\frac{y_2 f}{W[y_1, y_2]} = -\frac{t^{-1}(-6)}{-3} = -2t^{-1} \implies v_1 = \int -2t^{-1}\,\mathrm{d}t = -2\ln t$$
$$v_2' = \frac{y_1 f}{W[y_1, y_2]} = \frac{t^2(-6)}{-3} = 2t^2 \implies v_2 = \int 2t^2\,\mathrm{d}t = \frac{2}{3}t^3$$
$$y_p = -2\ln t(t^2) + \frac{2}{3}t^3(1/t) = \frac{2}{3}t^2 - 2t^2\ln t$$

The general solution is thus $y = y_h + y_p = c_1 t^2 + \dfrac{c_2}{t} + \dfrac{2}{3}t^2 - 2t^2\ln t$.

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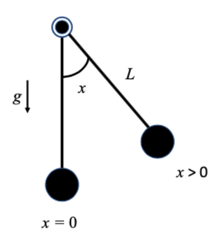
2. [2360/111721 (16 pts)] A simple, damped pendulum of length $L > 0$ rotates/oscillates about a fixed point with damping constant $\alpha \geq 0$. Let $x(t)$ denote the angle of the pendulum at time t (see figure).



When the amplitude of the oscillation is small and $g > 0$ is the acceleration due to gravity, we model the pendulum position by the equation

$$\ddot{x} + \frac{\alpha}{L}\dot{x} + \frac{g}{L}x = 0 \tag{1}$$

   (a) (3 pts) What is the pendulum's circular frequency (ω_0) in the absence of damping $(\alpha = 0)$?

   (b) (7 pts) Suppose the damping, α, has a fixed known value.

      i. (5 pts) Find all values of L such that the pendulum is underdamped.

      ii. (2 pts) For these value(s) of L will the pendulum pass through the equilibrium $(x = 0)$ more than once?

   (c) (6 pts) Determine the general solution of equation (1) with $\alpha = 2$, $g = 1$ and $L = 2$.

SOLUTION:

(a) The circular frequency is $\omega_0 = \sqrt{g/L}$.

(b) i. To be underdamped, we need

$$\frac{\alpha^2}{L^2} - 4(1)\frac{g}{L} < 0 \implies \alpha^2 < 4gL \implies L > \frac{\alpha^2}{4g}$$

ii. For these values of L, the pendulum will pass through the equilibrium more than once (infinitely many times).

(c) With these values of the parameters, the equation is $\ddot{x} + \dot{x} + \frac{1}{2}x = 0$. The characteristic equation is $r^2 + r + \frac{1}{2} = 0$ with roots

$$r = \frac{-1 \pm \sqrt{1 - 4(1)(\frac{1}{2})}}{2} = -\frac{1}{2} \pm \frac{1}{2}i \implies \alpha = -\frac{1}{2},\, \beta = \frac{1}{2}$$

so the general solution is $x(t) = e^{-t/2}\left[c_1\cos(t/2) + c_2\sin(t/2)\right]$.

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3. [2360/111721 (18 pts)] The following parts (a) and (b) are not related.

(a) (10 pts) The differential equation $2\ddot{x} + b\dot{x} + kx = F_0\cos(2\pi t)$ governs the motion of a certain harmonic oscillator.

i. (5 pts) For what value(s) of F_0, b and k, if any, will solutions to the equation grow without bound?

ii. (5 pts) Let $k = 0.25$ and $F_0 = 0$. Find all values of b (if any) that guarantee the solutions to the differential equation pass through the t-axis at most once.

(b) (8 pts) Convert the initial value problem $y^{(4)} + y'' = -\sin t,\ y(0) = 2,\ y'(0) = y''(0) = y'''(0) = 1$, to a system of first order differential equations with an appropriate initial condition. Write your final answer using matrices and vectors, if possible.

SOLUTION:

(a) i. The oscillator must be forced, so $F_0 \neq 0$. We need $b = 0$ (undamped) and $\omega_0 = \sqrt{\frac{k}{2}} = 2\pi \implies k = 8\pi^2$.

ii. We want the system to be critically damped or overdamped so $b^2 - 4(2)(0.25) \geq 0 \implies b^2 \geq 2 \implies b \geq \sqrt{2}$.

(b) Let $x_1 = y, x_2 = y', x_3 = y'', x_4 = y'''$. Then

$$\begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = x_3 \\ x_3' &= y''' = x_4 \\ x_4' &= y^{(4)} = -y'' - \sin t = -x_3 - \sin t \end{aligned}$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\sin t \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

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4. [2360/111721 (24 pts)] The following problems are not related.

(a) (6 pts) If $\mathscr{L}\{\cosh bt\} = \frac{s}{s^2 - b^2}$, use the transform table at the end of the exam to find $\mathscr{L}\{t\cosh bt\}$.

(b) (6 pts) Compute $\mathscr{L}^{-1}\left\{\frac{s+5}{s^2+4s+13}\right\}$.

(c) (12 pts) Use Laplace transforms to solve $y'' + 5y = 0,\ y(0) = 7,\ y'(0) = 1$.

SOLUTION:

(a)

$$\mathscr{L}\{t\cosh bt\} = (-1)^1\frac{\mathrm{d}}{\mathrm{d}s}\mathscr{L}\{\cosh bt\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s}{s^2 - b^2}\right) = \frac{s^2 + b^2}{(s^2 - b^2)^2}$$

(b) Complete the square in the denominator.

$$\mathscr{L}^{-1}\left\{\frac{s+5}{s^2+4s+13}\right\} = \mathscr{L}^{-1}\left\{\frac{s+5}{(s+2)^2+9}\right\} = \mathscr{L}^{-1}\left\{\frac{s+2}{(s+2)^2+9} + \frac{3}{(s+2)^2+9}\right\} = e^{-2t}\cos 3t + e^{-2t}\sin 3t$$

(c) With $Y(s) = \mathscr{L}\{y\}$ we have

$$s^2Y(s) - sy(0) - y'(0) + 5Y(s) = 0$$

$$\left(s^2+5\right)Y(s) = 7s+1$$

$$Y(s) = \frac{7s}{s^2+5} + \frac{1}{s^2+5} = \frac{7s}{s^2+5} + \frac{1}{\sqrt{5}}\frac{\sqrt{5}}{s^2+5}$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\left\{\frac{7s}{s^2+5} + \frac{1}{\sqrt{5}}\frac{\sqrt{5}}{s^2+5}\right\} = 7\cos\sqrt{5}t + \frac{\sqrt{5}}{5}\sin\sqrt{5}t$$

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5. [2360/111721 (22 pts)] A basis for the solution space of a certain third order homogeneous, constant coefficient linear differential equation $L(\vec{\mathbf{y}}) = 0$ is $\left\{1, t, e^{-2t}\right\}$.

(a) (3 pts) Write the characteristic equation.

(b) (3 pts) What is the general solution of the differential equation?

(c) (16 pts) Now consider the nonhomogeneous differential equation $L(\vec{\mathbf{y}}) = f(t)$. For each $f(t)$ below, write down the form of the particular solution you would use to solve the nonhomogeneous equation using the method of undetermined coefficients but **do not** solve for the coefficients.

i. $f(t) = 2$

ii. $f(t) = 3t^2 - 1$

iii. $f(t) = te^{2t}$

iv. $f(t) = \sin t + \cos 2t + e^t$

SOLUTION:

(a) $r^2(r+2) = 0$

(b) $y = c_1 + c_2t + c_3e^{-2t}$

(c) i. $y_p = At^2$

ii. $y_p = At^4 + Bt^3 + Ct^2$

iii. $y_p = (At+B)e^{2t}$

iv. $y_p = A\sin t + B\cos t + C\sin 2t + D\cos 2t + Ee^t$

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