**APPM 2360** **Exam 3** **Fall 2021**

- This exam is worth 100 points and has 5 questions.
- **Show all work and simplify your answers!** Answers with no justification will receive no points unless otherwise noted.
- **Please begin each problem on a new page.**
- **DO NOT** leave the exam until you have satisfactorily scanned and uploaded your exam to Gradescope.
- You are taking this exam in a proctored and honor code enforced environment. **NO** calculators, cell phones, or other electronic devices or the internet are permitted. You are allowed one 8.5"$\times$ 11" crib sheet with writing on one side only.

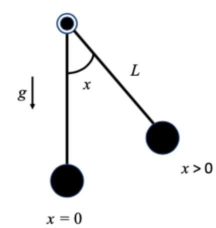
0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/111721 (20 pts)] Consider the differential equation $y'' - \frac{2}{t^2}y = -6$ for $t \geq 1$.

   (a) (10 pts) Show that the functions $y_1 = t^2$ and $y_2 = \frac{1}{t}$ form a basis for the solution space of the associated homogeneous problem. Justify your answer completely.

   (b) (10 pts) Find the general solution to the original nonhomogeneous problem.

2. [2360/111721 (16 pts)] A simple, damped pendulum of length $L > 0$ rotates/oscillates about a fixed point with damping constant $\alpha \geq 0$. Let $x(t)$ denote the angle of the pendulum at time $t$ (see figure).




When the amplitude of the oscillation is small and $g > 0$ is the acceleration due to gravity, we model the pendulum position by the equation

$$\ddot{x} + \frac{\alpha}{L}\dot{x} + \frac{g}{L}x = 0 \tag{1}$$

   (a) (3 pts) What is the pendulum's circular frequency ($\omega_0$) in the absence of damping ($\alpha = 0$)?

   (b) (7 pts) Suppose the damping, $\alpha$, has a fixed known value.

      i. (5 pts) Find all values of $L$ such that the pendulum is underdamped.

      ii. (2 pts) For these value(s) of $L$ will the pendulum pass through the equilibrium ($x = 0$) more than once?

   (c) (6 pts) Determine the general solution of equation (1) with $\alpha = 2$, $g = 1$ and $L = 2$.

3. [2360/111721 (18 pts)] The following parts (a) and (b) are not related.

   (a) (10 pts) The differential equation $2\ddot{x} + b\dot{x} + kx = F_0\cos(2\pi t)$ governs the motion of a certain harmonic oscillator.

      i. (5 pts) For what value(s) of $F_0, b$ and $k$, if any, will solutions to the equation grow without bound?

      ii. (5 pts) Let $k = 0.25$ and $F_0 = 0$. Find all values of $b$ (if any) that guarantee the solutions to the differential equation pass through the $t$-axis at most once.

   (b) (8 pts) Convert the initial value problem $y^{(4)} + y'' = -\sin t,\ y(0) = 2,\ y'(0) = y''(0) = y'''(0) = 1$, to a system of first order differential equations with an appropriate initial condition. Write your final answer using matrices and vectors, if possible.

4. [2360/111721 (24 pts)] The following problems are not related.

   (a) (6 pts) If $\mathscr{L}\{\cosh bt\} = \frac{s}{s^2 - b^2}$, use the transform table at the end of the exam to find $\mathscr{L}\{t\cosh bt\}$.

   (b) (6 pts) Compute $\mathscr{L}^{-1}\left\{\frac{s+5}{s^2+4s+13}\right\}$.

   (c) (12 pts) Use Laplace transforms to solve $y'' + 5y = 0,\ y(0) = 7,\ y'(0) = 1$.

**MORE PROBLEMS AND LAPLACE TRANSFORM TABLE ON NEXT PAGE / OTHER SIDE**

5. [2360/111721 (22 pts)] A basis for the solution space of a certain third order homogeneous, constant coefficient linear differential equation $L(\vec{\mathbf{y}}) = 0$ is $\{1, t, e^{-2t}\}$.

(a) (3 pts) Write the characteristic equation.

(b) (3 pts) What is the general solution of the differential equation?

(c) (16 pts) Now consider the nonhomogeneous differential equation $L(\vec{\mathbf{y}}) = f(t)$. For each $f(t)$ below, write down the form of the particular solution you would use to solve the nonhomogeneous equation using the method of undetermined coefficients but **do not** solve for the coefficients.

i. $f(t) = 2$

ii. $f(t) = 3t^2 - 1$

iii. $f(t) = te^{2t}$

iv. $f(t) = \sin t + \cos 2t + e^t$

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**Short table of Laplace Transforms:** $\mathscr{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, $a, b, c$ are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

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$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{tf'(t)\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\{f(t-c)\,\mathrm{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\mathrm{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$

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