1. [2360/102021 (15 pts)] Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

   (a) (6 pts) Show that the eigenvalues of $\mathbf{A}$ are $\lambda = 0$ and $\lambda = 2$.

   (b) (2 pts) State the algebraic multiplicity of each of the eigenvalues in part (a).

   (c) (4 pts) Find a basis for the eigenspace associated with $\lambda = 0$ and state its dimension.

   (d) (3 pts) Is it possible that the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$, where $\vec{\mathbf{b}} \neq \vec{\mathbf{0}}$, could be inconsistent? Explain briefly.

SOLUTION:

(a)

$$\begin{aligned}
\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} &= (1-\lambda)(-1)^{1+1}\begin{vmatrix} -\lambda & 0 \\ 0 & (1-\lambda) \end{vmatrix} + 1(-1)^{1+3}\begin{vmatrix} 0 & -\lambda \\ 1 & 0 \end{vmatrix} \\
&= (1-\lambda)^2(-\lambda) + \lambda \\
&= \lambda\left[1-(1-\lambda)^2\right] \\
&= \lambda\left[1-\left(1-2\lambda+\lambda^2\right)\right] \\
&= \lambda(2\lambda-\lambda^2) \\
&= \lambda^2(2-\lambda) = 0 \implies \lambda = 0, 2
\end{aligned}$$

(b) $\lambda = 0$ has algebraic multiplicity 2 whilst $\lambda = 2$ has algebraic multiplicity 1.

(c)

$$(\mathbf{A} - 0\mathbf{I})\,\vec{\mathbf{v}} = \vec{\mathbf{0}} \implies \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \implies \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \vec{\mathbf{v}} = \begin{bmatrix} -t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

A basis for the 2-dimensional eigenspace is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(d) Yes. Since 0 is an eigenvalue of the matrix $\mathbf{A}$, $|\mathbf{A}| = 0$ meaning that $\mathbf{A}$ is singular. The system will have either infinitely many solutions or none (inconsistent).

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2. [2360/102021 (18 pts)] The following problems are not related.

   (a) (6 pts) Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $\vec{\mathbf{x}} = \begin{bmatrix} x \\ y \end{bmatrix}$. Compute $\vec{\mathbf{x}}^{\mathrm{T}}\mathbf{A}\,\vec{\mathbf{x}}$.

   (b) (6 pts) If $\mathbf{C}^{-1} = \begin{bmatrix} 7 & -5 \\ 3 & 0 \end{bmatrix}$, $\mathbf{D}^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$ and $\vec{\mathbf{b}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, solve $\mathbf{CD}\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

   (c) (6 pts) Let $\mathbf{B} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 4 \\ 5 & 1 & -2 \end{bmatrix}$. Compute $(\operatorname{Tr}\mathbf{B})^2 - 4|\mathbf{B}|$ where $\operatorname{Tr}\mathbf{B}$ is the trace of $\mathbf{B}$.

SOLUTION:

(a)

$$\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} x+2y \\ 2x+y \end{bmatrix} = x^2 + 2xy + 2yx + y^2 = x^2 + 4xy + y^2$$

(b)

$$\mathbf{CD}\vec{\mathbf{x}} = \vec{\mathbf{b}} \implies \vec{\mathbf{x}} = (\mathbf{CD})^{-1}\vec{\mathbf{b}} = \mathbf{D}^{-1}\mathbf{C}^{-1}\vec{\mathbf{b}} = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 7 & -5 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$$

(c)

$$\operatorname{Tr}\mathbf{B} = 1 + (-2) + (-2) = -3$$

$$|\mathbf{B}| = \begin{vmatrix} 1 & 2 & -1 \\ 0 & -2 & 4 \\ 5 & 1 & -2 \end{vmatrix} = 1(-1)^{1+1}\begin{vmatrix} -2 & 4 \\ 1 & -2 \end{vmatrix} + 5(-1)^{3+1}\begin{vmatrix} 2 & -1 \\ -2 & 4 \end{vmatrix} = 1(0) + 5(6) = 30$$

$$(\operatorname{Tr}\mathbf{B})^2 - 4|\mathbf{B}| = 9 - 120 = -111$$

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3. [2360/102021 (18 pts)] The following problems are not related.

(a) (6 pts) Find the RREF of $\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 4 \\ 1 & 5 & 4 \\ 1 & 1 & -4 \end{bmatrix}$.

(b) (12 pts) Consider the linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ where the RREF of the augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 4 & 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

i. (5 pts) Find a particular solution to the nonhomogeneous system.

ii. (7 pts) Find a basis for the solution space of the associated homogeneous problem. What is its dimension?

SOLUTION:

(a)

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 4 \\ 1 & 5 & 4 \\ 1 & 1 & -4 \end{bmatrix} \xrightarrow[R_4^* = -1R_1 + R_4]{R_3^* = -1R_1 + R_3} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & -2 & -4 \end{bmatrix} \xrightarrow[R_2^* = \frac{1}{2}R_2]{R_3^* = -1R_2 + R_3,\ R_4^* = R_2 + R_4} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1^* = -3R_2 + R_1} \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) i. The free variables (those corresponding to the non-pivot columns) are x_2, x_4 with basic/lead variables (those corresponding to the pivot columns) being x_1, x_3, x_5. Setting $x_2 = s$ and $x_4 = t$ we have

$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 - 4s - 2t \\ s \\ 2 + t \\ t \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \\ -1 \end{bmatrix} + s\begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

A particular solution to the system of equations is, setting $s = t = 0$,

$$\vec{\mathbf{x}}_p = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \\ -1 \end{bmatrix}$$

ii. A basis for the solution space to the associated homogeneous problem is

$$\left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\},$$

the dimension of which is 2.

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4. [2360/102021 (20 pts)] Consider the functions $f_1 = 3t^2 + 2$, $f_2 = 4t - 6$ and $f_3 = 9t^2 - 2t + 9$ on the entire real line.

(a) (5 pts) Is $f_3 \in \text{span}\{f_1, f_2\}$? If so, write f_3 as a linear combination of f_1 and f_2. If not, explain why not.

(b) (5 pts) Compute $W[f_1, f_2](t)$. What can you conclude about the linear independence of $\{f_1, f_2\}$?

(c) (5 pts) Is $\{f_1, f_2\}$ a basis for $\mathbb{P}_2$, the space of all polynomials of degree less than or equal to 2? Why or why not?

(d) (5 pts) Is $\text{span}\{f_1, f_2, f_3\} = \mathbb{P}_2$? Why or why not?

SOLUTION:

(a) We need to see if there are constants c_1, c_2 such that $c_1 f_1 + c_2 f_2 = f_3$ or $c_1(3t^2 + 2) + c_2(4t - 6) = 9t^2 - 2t + 9$. Equating coefficients on either side of this equation yields the linear system

$$\begin{aligned} 3c_1 &= 9 \\ 4c_2 &= -2 \\ 2c_1 - 6c_2 &= 9 \end{aligned}$$

whose solution is $c_1 = 3, c_2 = -\frac{1}{2}$. Thus $f_3 = 3f_1 - \frac{1}{2}f_2$.

(b)

$$W[f_1, f_2](t) = \begin{vmatrix} 3t^2 + 2 & 4t - 6 \\ 6t & 4 \end{vmatrix} = 12t^2 + 8 - (24t^2 - 36t) = -12t^2 + 36t + 8 \neq 0$$

for at least one point on the real line. Thus, f_1 and f_2 are linearly independent on the real line.

(c) No. The dimension of $\mathbb{P}_2$ is 3 and we only have two linearly independent vectors.

(d) No. The set $\{f_1, f_2, f_3\}$ is linearly dependent so although there are three vectors in a space of dimension 3, the set cannot form a basis.

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5. [2360/102021 (15 pts)] Determine whether or not the subsets, $\mathbb{W}$, of the vector spaces, $\mathbb{V}$, are subspaces of $\mathbb{V}$. Justify your answers completely, assuming the standard operations of vector addition and scalar multiplication apply to each vector space.

(a) (5 pts) $\mathbb{V} = \mathcal{C}[0, 1]$ (the set of functions continuous on $[0, 1]$); $\mathbb{W} = \left\{ h(t) \in \mathcal{C}[0, 1] \,\middle|\, \int_0^1 h(t)\,\mathrm{d}t = 0 \right\}$.

(b) (5 pts) $\mathbb{V} = \mathbb{M}_{23}$; $\mathbb{W} = \left\{ \mathbf{A} \in \mathbb{M}_{23} \,\middle|\, \mathbf{A} = \begin{bmatrix} a & a^2 & a^3 \\ b^3 & b & 1 \end{bmatrix}, a, b \in \mathbf{R} \right\}$.

(c) (5 pts) $\mathbb{V} = \mathbb{R}^2$; $\mathbb{W} = \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 \leq 9 \right\}$.

SOLUTION:

(a) Since the zero vector is in the subset $\left(\int_0^1 0\,\mathrm{d}t = 0 \right)$ we need to check for closure. We'll use linear combinations. To that end, assume $f \in \mathbb{W}$, that is, $\int_0^1 f(t)\,\mathrm{d}t = 0$ and $g \in \mathbb{W}$, that is, $\int_0^1 g(t)\,\mathrm{d}t = 0$. and let $a, b \in \mathbb{R}$. Then

$$\int_0^1 (af + bg)(t)\,\mathrm{d}t = a \int_0^1 f(t)\,\mathrm{d}t + b \int_0^1 g(t)\,\mathrm{d}t = a \cdot 0 + b \cdot 0 = 0 \implies af + bg \in \mathbb{W}$$

showing that $\mathbb{W}$ is closed with respect to linear combinations and thus is a subspace of $\mathbb{V}$.

(b) $\vec{\mathbf{0}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \notin \mathbb{W} \implies \mathbb{W}$ is not a subspace.

(c) Any one of the following arguments is sufficient

i. $(1, 1) \in \mathbb{W}$ since $1^2 + 1^2 \leq 9$. However, $3(1, 1) = (3, 3) \notin \mathbb{W}$ since $3^2 + 3^2 = 18 > 9 \implies \mathbb{W}$ not closed under scalar multiplication, implying that $\mathbb{W}$ is not a subspace.

ii. $(1, 1) \in \mathbb{W}$ (as above) and $(2, 2) \in \mathbb{W}$ since $2^2 + 2^2 = 8 \leq 9$. However, $(1, 1) + (2, 2) = (3, 3) \notin \mathbb{W}$, again, since $3^2 + 3^2 = 18 > 9 \implies \mathbb{W}$ not closed under vector addition, implying that $\mathbb{W}$ is not a subspace.

iii. $(1, 1) \in \mathbb{W}$, $(2, 2) \in \mathbb{W}$ but $2(1, 1) + 4(2, 2) = (9, 9) \notin \mathbb{W}$ since $9^2 + 9^2 = 162 > 9 \implies \mathbb{W}$ not closed under linear combinations, implying that $\mathbb{W}$ is not a subspace.

6. [2360/102021 (14 pts)] If $\mathbf{A}$ is a 5×5 matrix with $|\mathbf{A}| = -2$, $\mathbf{B}$ is a 6×5 matrix and p, q are nonzero scalars, compute the following, if possible, or state that it can be determined, or state that it is undefined or cannot be determined. No justification required. No partial credit awarded.

(a) $\mathbf{B}^2$

(b) $|\mathbf{A}^3|$

(c) The order/size of $\mathbf{B}\mathbf{B}^{\mathrm{T}}$

(d) $|2\mathbf{A}|$

(e) $p\mathbf{B} + q\mathbf{B}^{\mathrm{T}}$

(f) Solve the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ for any given column vector $\vec{\mathbf{b}}$.

(g) Solve $\mathbf{B}\vec{\mathbf{y}} = \vec{\mathbf{c}}$ by Cramer's Rule for any column vector $\vec{\mathbf{c}}$.

SOLUTION:

(a) $\mathbf{B}^2 = \mathbf{B}\mathbf{B}$ which is undefined: $(6 \times 5)(6 \times 5)$ cannot be done

(b) $|\mathbf{A}^3| = |\mathbf{A}\mathbf{A}\mathbf{A}| = |\mathbf{A}||\mathbf{A}||\mathbf{A}| = (-2)(-2)(-2) = -8$

(c) $(6 \times 5)(5 \times 6)$ gives an order of 6×6.

(d) $|2\mathbf{A}| = 2^5(-2) = -64$

(e) $\mathbf{B}$ and $\mathbf{B}^{\mathrm{T}}$ do not have have the same order and thus cannot be added for any $p, q \neq 0$

(f) $\mathbf{A}$ is invertible so $\vec{\mathbf{x}} = \mathbf{A}^{-1}\vec{\mathbf{b}}$

(g) Cramer's Rule can only be used on square systems