1. [2360/102021 (15 pts)] Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

   (a) (6 pts) Show that the eigenvalues of $\mathbf{A}$ are $\lambda = 0$ and $\lambda = 2$.

   (b) (2 pts) State the algebraic multiplicity of each of the eigenvalues in part (a).

   (c) (4 pts) Find a basis for the eigenspace associated with $\lambda = 0$ and state its dimension.

   (d) (3 pts) Is it possible that the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$, where $\vec{\mathbf{b}} \neq \vec{\mathbf{0}}$, could be inconsistent? Explain briefly.

2. [2360/102021 (18 pts)] The following problems are not related.

   (a) (6 pts) Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $\vec{\mathbf{x}} = \begin{bmatrix} x \\ y \end{bmatrix}$. Compute $\vec{\mathbf{x}}^{\mathrm{T}}\mathbf{A}\vec{\mathbf{x}}$.

   (b) (6 pts) If $\mathbf{C}^{-1} = \begin{bmatrix} 7 & -5 \\ 3 & 0 \end{bmatrix}$, $\mathbf{D}^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$ and $\vec{\mathbf{b}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, solve $\mathbf{CD}\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

   (c) (6 pts) Let $\mathbf{B} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 4 \\ 5 & 1 & -2 \end{bmatrix}$. Compute $(\mathrm{Tr}\,\mathbf{B})^2 - 4|\mathbf{B}|$ where $\mathrm{Tr}\,\mathbf{B}$ is the trace of $\mathbf{B}$.

3. [2360/102021 (18 pts)] The following problems are not related.

   (a) (6 pts) Find the RREF of $\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 4 \\ 1 & 5 & 4 \\ 1 & 1 & -4 \end{bmatrix}$.

   (b) (12 pts) Consider the linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ where the RREF of the augmented matrix is

   $$\left[\begin{array}{ccccc|c} 1 & 4 & 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

      i. (5 pts) Find a particular solution to the nonhomogeneous system.

      ii. (7 pts) Find a basis for the solution space of the associated homogeneous problem. What is its dimension?

4. [2360/102021 (20 pts)] Consider the functions $f_1 = 3t^2 + 2$, $f_2 = 4t - 6$ and $f_3 = 9t^2 - 2t + 9$ on the entire real line.

   (a) (5 pts) Is $f_3 \in \mathrm{span}\,\{f_1, f_2\}$? If so, write f_3 as a linear combination of f_1 and f_2. If not, explain why not.

   (b) (5 pts) Compute $W[f_1, f_2](t)$. What can you conclude about the linear independence of $\{f_1, f_2\}$?

   (c) (5 pts) Is $\{f_1, f_2\}$ a basis for $\mathbb{P}_2$, the space of all polynomials of degree less than or equal to 2? Why or why not?

   (d) (5 pts) Is $\mathrm{span}\,\{f_1, f_2, f_3\} = \mathbb{P}_2$? Why or why not?

5. [2360/102021 (15 pts)] Determine whether or not the subsets, $\mathbb{W}$, of the vector spaces, $\mathbb{V}$, are subspaces of $\mathbb{V}$. Justify your answers completely, assuming the standard operations of vector addition and scalar multiplication apply to each vector space.

   (a) (5 pts) $\mathbb{V} = \mathcal{C}[0,1]$ (the set of functions continuous on $[0,1]$); $\mathbb{W} = \left\{ h(t) \in \mathcal{C}[0,1] \,\middle|\, \int_0^1 h(t)\,\mathrm{d}t = 0 \right\}$.

   (b) (5 pts) $\mathbb{V} = \mathbb{M}_{23}$; $\mathbb{W} = \left\{ \mathbf{A} \in \mathbb{M}_{23} \,\middle|\, \mathbf{A} = \begin{bmatrix} a & a^2 & a^3 \\ b^3 & b & 1 \end{bmatrix}, a, b \in \mathbf{R} \right\}$.

   (c) (5 pts) $\mathbb{V} = \mathbb{R}^2$; $\mathbb{W} = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 \leq 9 \right\}$.

6. [2360/102021 (14 pts)] If $\mathbf{A}$ is a 5×5 matrix with $|\mathbf{A}| = -2$, $\mathbf{B}$ is a 6×5 matrix and p, q are nonzero scalars, compute the following, if possible, or state that it can be determined, or state that it is undefined or cannot be determined. No justification required. No partial credit awarded.

   (a) $\mathbf{B}^2$

(b) $|\mathbf{A}^3|$

(c) The order/size of $\mathbf{B}\mathbf{B}^{\mathrm{T}}$

(d) $|2\mathbf{A}|$

(e) $p\mathbf{B} + q\mathbf{B}^{\mathrm{T}}$

(f) Solve the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ for any given column vector $\vec{\mathbf{b}}$.

(g) Solve $\mathbf{B}\vec{\mathbf{y}} = \vec{\mathbf{c}}$ by Cramer's Rule for any column vector $\vec{\mathbf{c}}$.