1. [2360/092221 (25 pts)] Consider the differential equation $y' = \dfrac{a}{bt^2 + y^c + 2}$ where a, b, c represent real numbers and $c \neq 0$.

   (a) (9 pts) Find the value(s) of a, b, and $c \neq 0$, if any, that make the equation:

   i. (3 pts) autonomous

   ii. (3 pts) linear

   iii. (3 pts) separable

   (b) (16 pts) For the following four questions, set $a = 6$, $b = 1$, and $c = 2$ in the differential equation above.

   i. (4 pts) Graph the isocline corresponding to a slope of 1. Include line segments on the isocline indicating the slope of the solution there.

   ii. (4 pts) Does the equation possess solutions that are decreasing anywhere? Explain briefly.

   iii. (4 pts) For what values, if any, of (t_0, y_0) does Picard's theorem guarantee the existence of a unique solution to the differential equation passing through the point (t_0, y_0)? Justify your answer.

   iv. (4 pts) Use a single step of Euler's method to estimate the value of the solution passing through $(0, 1)$ at the point $t = 0.1$.

SOLUTION:

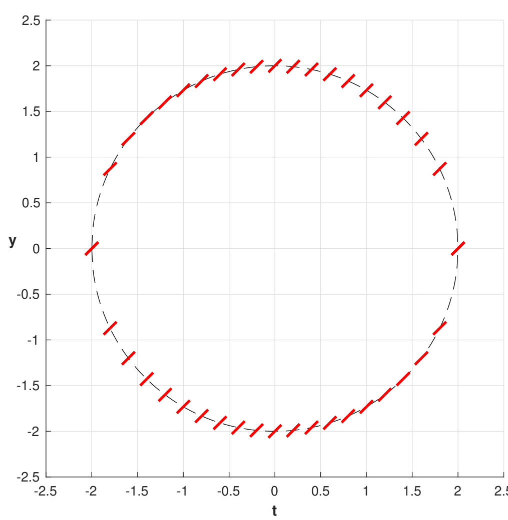
(a) i. $b = 0$, a, c arbitrary ; or $a = 0$, b, c arbitrary

ii. $a = 0$, b, c arbitrary

iii. $b = 0$, a, c arbitrary ; or $a = 0$, b, c arbitrary

(b) i. We have $y' = \dfrac{6}{t^2 + y^2 + 2}$ so to find the isocline where the slope is 1 set $y' = 1$ giving

$$\frac{6}{t^2 + y^2 + 2} = 1 \implies t^2 + y^2 = 4$$



ii. No. Since $\dfrac{6}{t^2 + y^2 + 2} > 0$ for all (t, y), solution curves are increasing everywhere.

iii. Since $f(t, y) = \dfrac{6}{t^2 + y^2 + 2}$ and $f_y(t, y) = \dfrac{-12y}{(t^2 + y^2 + 2)^2}$ are continuous throughout $\mathbb{R}^2$, Picard's theorem guarantees the existence of a unique solution for all values of (t_0, y_0).

iv. Euler's method gives

$$y_{n+1} = y_n + hf(t_n, y_n) = y_n + \frac{6h}{t_n^2 + y_n^2 + 2}$$

and with stepsize $h = 0.1$

$$y(0.1) \approx y_1 = y_0 + \frac{6(0.1)}{t_0^2 + y_0^2 + 2} = 1 + \frac{6(0.1)}{0^2 + 1^2 + 2} = \frac{6}{5} = 1.2$$

2. [2360/092221 (17 pts)] Consider the differential equation $y' + ty - 4ty^3 = 0$.

   (a) (3 pts) Find all equilibrium solutions of the equation.

   (b) (3 pts) Show that dividing the differential equation by y^3 and making the substitution $v = y^{-2}$ transforms the equation into the linear differential equation $v' - 2tv + 8t = 0$.

   (c) (8 pts) Use the integrating factor method to solve the linear equation from part (b).

   (d) (3 pts) Solve the initial value problem consisting of the original differential equation and the initial condition $y(0) = \frac{1}{4}$.

SOLUTION:

(a) Rewrite the differential equation as $y' = 4ty^3 - ty = ty(4y^2 - 1)$. Then $ty(4y^2 - 1) = 0 \implies y = 0, \pm\frac{1}{2}$ are the equilibrium solutions.

(b) If $v = y^{-2}$, then $v' = -2y^{-3}y'$ and

$$\begin{aligned} y^{-3}y' + ty^{-2} - 4t &= 0 \\ -\frac{1}{2}v' + tv - 4t &= 0 \\ v' - 2tv + 8t &= 0 \end{aligned}$$

where the last equation is the linear equation we seek.

(c) With $p(t) = -2t$, the integrating factor is

$$\mu(t) = e^{\int -2t\,\mathrm{d}t} = e^{-t^2}$$

Multiplying the differential equation by $\mu(t)$ and rearranging yields

$$\left(e^{-t^2}v\right)' = -8te^{-t^2}$$

which, after indefinite integration gives

$$e^{-t^2}v = 4e^{-t^2} + C \implies v = 4 + Ce^{t^2}$$

(d) Transforming the solution in part (b) back to the original dependent variable y we have

$$v = y^{-2} = 4 + Ce^{t^2} \implies y = \frac{\pm 1}{\sqrt{4 + Ce^{t^2}}}$$

and applying the initial condition yields

$$y(0) = \frac{1}{4} = \frac{\pm 1}{\sqrt{4 + C}} \implies C = 12$$

giving

$$y = \frac{1}{2\sqrt{1 + 3e^{t^2}}}$$

as the solution to the initial value problem. Note that we had to drop the $\pm$ since only the positive solution satisfies the initial value.

3. [2360/092221 (14 pts)] A 200 liter coffee pot is initially three quarters full of pure coffee. Coffee containing $1/(t+1)$ grams of sugar per liter (t is time) is poured into the pot at a rate of 4 liters per minute. The well-mixed sweetened coffee is drained from the pot at a rate of 6 liters per minute.

   (a) (8 pts) Write, but do not solve, the governing initial value problem. Be sure to identify your variables.

   (b) (4 pts) Fully classify (order, linearity, homogeneity, type of coefficient) the differential equation.

   (c) (2 pts) Based on the physical situation, over what time interval is the equation valid?

SOLUTION:

(a) Let t be the time in minutes, $x(t)$ be the amount (grams) of sugar at time t and $V(t)$ be the volume (liters) of sweetened coffee in the tank at time t. We begin by finding the volume of sweetened coffee in the pot at any time, noting that initially there are 150 liters in the pot. This gives the initial value problem with solution

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \text{flow rate in} - \text{flow rate out} = 4 - 6 = -2,\ V(0) = 150$$

$$\int \mathrm{d}V = \int -2\,\mathrm{d}t$$

$$V(t) = -2t + C$$

$$V(0) = -2(0) + C = 150$$

$$V(t) = 150 - 2t$$

Next we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \text{rate in} - \text{rate out} = \left(\frac{1}{t+1}\frac{\text{gram}}{\text{liter}}\right)\left(4\frac{\text{liter}}{\text{minute}}\right) - \left(\frac{x}{150-2t}\frac{\text{gram}}{\text{liter}}\right)\left(6\frac{\text{liter}}{\text{minute}}\right)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{3x}{75-t} = \frac{4}{t+1},\ x(0) = 0$$

where the initial condition comes from the fact that there is no sugar in the pot at the initial time.

(b) First order, linear, nonhomogeneous, variable coefficient

(c) After 75 minutes, the coffee pot will be empty. The equation is valid for $0 \le t \le 75$.

4. [2360/092221 (14 pts)] Use the Euler-Lagrange two stage method (variation of parameters) to solve the initial value problem

$$ty' + 4y = \frac{\cos t}{t^2},\ y(\pi) = 2/\pi^4, \quad (t > 0)$$

SOLUTION:

Solve the associated homogeneous equation using separation of variables:

$$t\frac{\mathrm{d}y_h}{\mathrm{d}t} = -4y_h$$

$$\int \frac{\mathrm{d}y_h}{y_h} = -\int \frac{4}{t}\,\mathrm{d}t$$

$$\ln|y_h| = -4\ln|t| + k$$

$$|y_h| = e^k|t|^{-4}$$

$$y = Ct^{-4}, \quad C \in \mathbb{R}$$

Let $y_p = v(t)t^{-4}$ and substitute this into the nonhomogeneous equation to get

$$ty_p' + 4y_p = t\left(-4vt^{-5} + v't^{-4}\right) + 4vt^{-4} = \frac{\cos t}{t^2}$$

$$v't^{-3} = \frac{\cos t}{t^2}$$

$$\int v'\,\mathrm{d}t = \int t\cos t\,\mathrm{d}t \qquad \text{integration by parts}$$

$$v = t\sin t + \cos t$$

$$\implies y_p = (t\sin t + \cos t)\,t^{-4}$$

Apply the Nonhomogeneous Principle, yielding

$$y = y_h + y_p = t^{-4}(C + t\sin t + \cos t)$$

Using the initial condition gives

$$y(\pi) = \frac{C-1}{\pi^4} = \frac{2}{\pi^4} \implies C = 3$$

so that the final solution is $y = t^{-4}(3 + t\sin t + \cos t)$.

5. [2360/092221 (14 pts)] Suppose that a certain fish population, $z > 0$, changes at a rate given by the logistic-like equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} = (1 - z^2)z$$

Find the general solution of this differential equation. Leave your answer in implicit form without natural logarithms.

SOLUTION:

Use separation of variables and partial fractions.

$$\frac{\mathrm{d}z}{z(1+z)(1-z)} = \mathrm{d}t$$

$$\int \left(\frac{1}{z} + \frac{1/2}{1-z} - \frac{1/2}{1+z}\right) \mathrm{d}z = \int dt$$

$$\ln|z| - \frac{1}{2}\ln|1-z| - \frac{1}{2}\ln|1+z| = t + k$$

$$\ln|z| - \frac{1}{2}\ln|(1-z)(1+z)| = t + k$$

$$\ln|z| - \frac{1}{2}\ln|1-z^2| = t + k$$

$$\ln\left|\frac{z}{\sqrt{1-z^2}}\right| = t + k$$

$$\frac{z}{\sqrt{1-z^2}} = Ce^t, \quad C \in \mathbb{R}$$

6. [2360/092221 (16 pts)] Consider the system of differential equations

$$x' = y^2 - 4$$
$$y' = e^x - 5$$

(a) (4 pts) Find the h nullclines, if any.

(b) (4 pts) Find the v nullclines, if any.

(c) (4 pts) Find all equilibrium solutions, if any.

(d) (4 pts) Plot the element of the vector field at the origin.

SOLUTION:

(a) h nullclines occur where $y' = e^x - 5 = 0 \implies x = \ln 5$.

(b) v nullclines occur where $x' = y^2 - 4 = 0 \implies y = \pm 2$.

(c) Equilibrium solutions occur where the v and h nullclines intersect or where x' and y' both vanish. These are $(\ln 5, 2)$ and $(\ln 5, -2)$.

(d) The vector field is given by $\mathbf{V}(x, y) = \langle y^2 - 4, e^x - 5\rangle$ so that $\mathbf{V}(0, 0) = \langle -4, -4\rangle$.