1. [2360/092221 (25 pts)] Consider the differential equation $y' = \dfrac{a}{bt^2 + y^c + 2}$ where a, b, c represent real numbers and $c \neq 0$.

   (a) (9 pts) Find the value(s) of a, b, and $c \neq 0$, if any, that make the equation:

      i. (3 pts) autonomous

      ii. (3 pts) linear

      iii. (3 pts) separable

   (b) (16 pts) For the following four questions, set $a = 6$, $b = 1$, and $c = 2$ in the differential equation above.

      i. (4 pts) Graph the isocline corresponding to a slope of 1. Include line segments on the isocline indicating the slope of the solution there.

      ii. (4 pts) Does the equation possess solutions that are decreasing anywhere? Explain briefly.

      iii. (4 pts) For what values, if any, of (t_0, y_0) does Picard's theorem guarantee the existence of a unique solution to the differential equation passing through the point (t_0, y_0)? Justify your answer.

      iv. (4 pts) Use a single step of Euler's method to estimate the value of the solution passing through $(0, 1)$ at the point $t = 0.1$.

2. [2360/092221 (17 pts)] Consider the differential equation $y' + ty - 4ty^3 = 0$.

   (a) (3 pts) Find all equilibrium solutions of the equation.

   (b) (3 pts) Show that dividing the differential equation by y^3 and making the substitution $v = y^{-2}$ transforms the equation into the linear differential equation $v' - 2tv + 8t = 0$.

   (c) (8 pts) Use the integrating factor method to solve the linear equation from part (b).

   (d) (3 pts) Solve the initial value problem consisting of the original differential equation and the initial condition $y(0) = \frac{1}{4}$.

3. [2360/092221 (14 pts)] A 200 liter coffee pot is initially three quarters full of pure coffee. Coffee containing $1/(t + 1)$ grams of sugar per liter (t is time) is poured into the pot at a rate of 4 liters per minute. The well-mixed sweetened coffee is drained from the pot at a rate of 6 liters per minute.

   (a) (8 pts) Write, but do not solve, the governing initial value problem. Be sure to identify your variables.

   (b) (4 pts) Fully classify (order, linearity, homogeneity, type of coefficient) the differential equation.

   (c) (2 pts) Based on the physical situation, over what time interval is the equation valid?

4. [2360/092221 (14 pts)] Use the Euler-Lagrange two stage method (variation of parameters) to solve the initial value problem

$$ty' + 4y = \frac{\cos t}{t^2}, \; y(\pi) = 2/\pi^4, \quad (t > 0)$$

5. [2360/092221 (14 pts)] Suppose that a certain fish population, $z > 0$, changes at a rate given by the logistic-like equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} = (1 - z^2)z$$

   Find the general solution of this differential equation. Leave your answer in implicit form without natural logarithms.

6. [2360/092221 (16 pts)] Consider the system of differential equations

$$x' = y^2 - 4$$
$$y' = e^x - 5$$

   (a) (4 pts) Find the h nullclines, if any.

   (b) (4 pts) Find the v nullclines, if any.

   (c) (4 pts) Find all equilibrium solutions, if any.

   (d) (4 pts) Plot the element of the vector field at the origin.

APPM 2360 Exam
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