1. [APPM 2360 Exam (25 pts)] The following problems (a)-(d) are not related.

(a) (11 pts) Consider the linear system $A\vec{x} = \vec{b}$. After several elementary row operations, the augmented matrix of the system is

\[
\begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0
\end{bmatrix}
\]

i. (2 pts) Is the matrix in RREF? If it is not, make it so.

ii. (3 pts) Find a particular solution to the linear system.

iii. (3 pts) Find a basis for and the dimension of the solution space to the associated homogeneous problem.

iv. (3 pts) Use the Nonhomogeneous Principle to write the general solution of the nonhomogeneous system.

(b) (5 pts) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, compute $A^2 - 5A - 2I$.

(c) (5 pts) Is span \(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \end{bmatrix}\) = $\mathbb{R}^3$? Justify your answer.

(d) (4 pts) Let $W$ be the set of matrices of the form $\begin{bmatrix} a & b & c \\ p & q & r \end{bmatrix}$ where $b, c, p, q, r$ are real numbers and $a$ is an integer. Is $W$ a subspace of $M_{23}$? Justify your answer.

**SOLUTION:**

(a)

i. No. RREF is

\[
\begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0
\end{bmatrix}
\]

ii. From the RREF in part (i) we have $x_1 = 1 - x_3, x_2 = -x_3$ with $x_3$ as free variable so that with $r$ a real number

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix} 1 - r \\ -r \\ r \end{bmatrix}
\]

A particular solution is obtained by letting $r = 0$ or $\vec{x}_p = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

iii. The RREF of the augmented matrix corresponding to the associated homogeneous system is

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

implying that $x_1 = -x_3, x_2 = -x_3$ with $x_3$ a free variable. With $r$ a real number we have $\vec{x}_h = r \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ so that a basis for the solution space of the associated homogeneous system is \(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\) which has dimension 1.

iv. The Nonhomogeneous Principle then gives the general solution to the system as

\[
\vec{x} = r \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad r \in \mathbb{R}
\]

(b)

\[
A^2 - 5A - 2I = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 5 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]
(c) Since the dimension of $\mathbb{R}^3$ is three, there are enough vectors to potentially be a basis, but they must be linearly independent to actually be a basis. To check this, we need to see if $c_1 = c_2 = c_3 = 0$ is the only solution to

$$
c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} -5 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

which is equivalent to seeing if the trivial solution is the sole solution to

$$
\begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & -4 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

Since

$$
\begin{vmatrix} 1 & 2 & -5 \\ 0 & 1 & -4 \\ -1 & -1 & 1 \end{vmatrix} = 0
$$

there exist nontrivial solutions to the aforementioned linear system, implying that the vectors are linearly dependent and thus do not form a basis for $\mathbb{R}^3$. Hence the span of the vectors is not $\mathbb{R}^3$.

(d) Let $A = \begin{bmatrix} a & b & c \\ p & q & r \end{bmatrix} \in \mathcal{W}$ and let $t$ be a real number. Then $tA = \begin{bmatrix} ta & tb & tc \\ tp & tq & tr \end{bmatrix} \notin \mathcal{W}$ since $ta$ is a real number, not necessarily an integer. Thus $\mathcal{W}$ is not closed under scalar multiplication and is not a subspace of $M_{23}$.

2. [APPM 2360 Exam (25 pts)] The following problems (a)-(c) are not related.

(a) (6 pts) Which of the following differential equations have a stable equilibrium solution at $y(t) = 1$? Briefly justify your answer.
   i. $y' = -y$
   ii. $y' = y(y - 1)$
   iii. $y' = (y - 1)(y - 2)$

(b) (12 pts) Consider the system

$$
x' = \left( x - \frac{1}{2} \right) x - xy
\quad y' = (x - 1) y
$$

i. (6 pts) Find all equilibrium points of the system.

ii. (6 pts) Plot and appropriately label all $h$ and $v$ nullclines in the phase plane. At the points $(1, 1)$ and $\left( \frac{3}{2}, 1 \right)$ draw arrows indicating the direction that the trajectories through the nullclines would take at those points.

(c) (7 pts) Solve the initial value problem $y' = \sqrt{ty}, \; y(1) = 4/9$. Write your solution as an explicit function of $t$.

**SOLUTION:**

(a) i. $y(t) = 1$ is not an equilibrium solution.

ii. $y(t) = 1$ is an unstable equilibrium solution as noted below.

$$
y > 1: \quad y' > 0
\quad 0 < y < 1: \quad y' < 0
\quad y < 0: \quad y' > 0
$$

iii. $y(t) = 1$ is a stable equilibrium solution as noted below.

$$
y > 2: \quad y' > 0
\quad 1 < y < 2: \quad y' < 0
\quad y < 1: \quad y' > 0
$$
(b) i. We need to solve the following nonlinear system of simultaneous equations:

\[
\begin{align*}
\left( x - \frac{1}{2} \right) x - xy &= 0 \\
(x - 1) y &= 0
\end{align*}
\]

If \( y = 0 \) in (2), this forces \( x = \frac{1}{2}, 0 \) in (1). If \( x = 1 \) in (2), then \( y = \frac{1}{2} \) in (1). The equilibrium points are thus \((\frac{1}{2}, 0), (0, 0), (1, \frac{1}{2})\).

ii. \( v \) nullclines: \( x' = \left( x - \frac{1}{2} \right) x - xy = 0 \implies x = 0, y = x - \frac{1}{2} \) (red)

\( h \) nullclines: \( y' = (x - 1) y = 0 \implies y = 0, x = 1 \) (blue)

(c) Use separation of variables.

\[
\int y^{-1/2} dy = \int t^{1/2} dt \quad \implies \quad 2y^{1/2} = \frac{2}{3} t^{3/2} + C \quad \text{apply initial condition}
\]

\[
2 \left( \frac{4}{9} \right)^{1/2} = \frac{2}{3} 1^{3/2} + C \quad \implies \quad C = \frac{2}{3}
\]

\[
2y^{1/2} = \frac{2}{3} t^{3/2} + 1 \quad \implies \quad y = \frac{1}{9} \left( t^{3/2} + 1 \right)^2
\]

3. [APPM 2360 Exam (40 pts)] The following problems (a)-(d) are not related.

(a) (12 pts) Find the general solution of \( y'' - 4y' + 4y = \frac{3e^{2t}}{t} \) on the interval \( t > 0 \). Look carefully at the equation before deciding how to solve it.

(b) (8 pts) A critically damped mass/spring harmonic oscillator has a restoring constant of 32 N/m and a \( \frac{1}{2} \) kg mass. To start the motion, the mass is released from rest 7 units to the left of the equilibrium position.

i. (4 pts) Set up, but do not solve, the initial value problem describing the unforced oscillator.

ii. (4 pts) Now suppose the aforementioned oscillator is driven with the forcing function in the following graph, where the wavy portion of the graph is a portion of \( 2 \sin \left( \frac{\pi}{2} t \right) \). Using step functions, how does the differential equation from part (i) change? Do not solve this new equation.

(c) (8 pts) Find \( \mathcal{L} \{ e^{2t} \text{ step}(t - 3) \} \).
(d) (12 pts) Solve the initial value problem \( y'' + 25y = \delta(t - 2), \ y(0) = 1, \ y'(0) = -5. \)

**SOLUTION:**

(a) The characteristic equation for the homogeneous equation is \( r^2 - 4r + 4 = (r - 2)^2 = 0 \) giving the solutions to the homogeneous equation as \( y_1 = e^{2t} \) and \( y_2 = te^{2t}. \) Using variation of parameters the particular solution takes the form \( y_p = v_1y_1 + v_2y_2. \) The Wronskian of the two solutions of the homogeneous equation is

\[
W[e^{2t}, te^{2t}] = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & 2te^{2t} + e^{2t} \end{vmatrix} = e^{4t}
\]

so that with \( f(t) = 3e^{2t}/t \) we have

\[
v' = -y_2 f \Rightarrow v_1 = -3t
\]

\[
v' = y_1 f \Rightarrow v_2 = 3 \ln |t| = 3 \ln t \quad \text{since} \ t > 0
\]

\[
y = y_p = c_1 e^{2t} + c_2 te^{2t} - 3te^{2t} + 3te^{2t} \ln t = c_1 e^{2t} + (c_2 - 3)te^{2t} + 3te^{2t} \ln t
\]

(b) i. Critically damped means that \( b^2 - 4mk = 0 \Rightarrow b^2 - 4 \left( \frac{1}{2} \right) (32) = 0 \Rightarrow b = 8. \) With \( x(t) \) representing the displacement of the mass,

\[
\frac{1}{2} \ddot{x} + 8 \dot{x} + 32x = 0, \quad x(0) = -7, \quad \dot{x}(0) = 0
\]

ii.

\[
\frac{1}{2} \ddot{x} + 8 \dot{x} + 32x = 2 - t - (2 - t) \text{step}(t - 2) + 2 \sin \left( \frac{\pi t}{2} \right) [\text{step}(t - 2) - \text{step}(t - 5)] + 2 \text{step}(t - 5)
\]

or

\[
= (2 - t) [1 - \text{step}(t - 2)] + 2 \sin \left( \frac{\pi t}{2} \right) [\text{step}(t - 2) - \text{step}(t - 5)] + 2 \text{step}(t - 5)
\]

or

\[
= (2 - t) [\text{step}(t) - \text{step}(t - 2)] + 2 \sin \left( \frac{\pi t}{2} \right) [\text{step}(t - 2) - \text{step}(t - 5)] + 2 \text{step}(t - 5)
\]

(c)

\[
\mathcal{L} \{ e^{2t} \text{step}(t - 3) \} = e^{-3s} \mathcal{L} \{ e^{2(t+3)} \} = e^{-3s} \mathcal{L} \{ e^{6} \} = \frac{e^{6-3s}}{s-2}
\]

(d) Use Laplace transforms.

\[
s^2 Y(s) - sy(0) - y'(0) + 25Y(s) = e^{-2s}
\]

\[
\left( s^2 + 25 \right) Y(s) = e^{-2s} + s - 5
\]

\[
Y(s) = \frac{e^{-2s}}{s^2 + 25} + \frac{s}{s^2 + 25} - \frac{5}{s^2 + 25}
\]

\[
y(t) = \frac{1}{5} \mathcal{L}^{-1} \left\{ \frac{5e^{-2s}}{s^2 + 25} \right\} + \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 25} \right\} - \mathcal{L}^{-1} \left\{ \frac{5}{s^2 + 25} \right\}
\]

\[
= \frac{1}{5} \sin 5t (t - 2) \text{step}(t - 2) + \cos 5t - \sin 5t
\]

4. [APPM 2360 Exam (35 pts)] The following problems (a)-(c) are not related.

(a) (15 pts) Solve the initial value problem \( \bar{x}' = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \bar{x}, \quad \bar{x}(0) = \begin{bmatrix} -1 \\ 8 \end{bmatrix} \). Write your answer as a single vector.

(b) (12 pts) Consider the system \( \bar{x}' = A \bar{x} \) where \( A = \begin{bmatrix} 2 & 1 \\ 0 & k \end{bmatrix} \) and \( k \) is a parameter.
i. (4 pts) Find \( \text{Tr } A \) and \( |A| \).

ii. (4 pts) Classify the geometry and stability properties of the system for the following values of \( k \).
   \[ \text{A. } k < 0 \quad \text{B. } k = 1 \]

iii. (4 pts) For \( k > 0 \) can the fixed point ever be a spiral or center? Briefly explain mathematically why or why not.

(c) (8 pts) Two tanks, each holding 100 gallons (G) of liquid, are interconnected by pipes, with the liquid flowing from tank 1 into tank 2 at a rate of 3 G/hour and from tank 2 into tank 1 at a rate of 1 G/hour. The liquid inside each tank is kept well stirred. A brine solution with a concentration of 2 kg/G of salt flows into tank 1 at a rate of 6 G/hour. The (diluted) solution flows out of the system from tank 1 at 4 G/hour and from tank 2 at 2 G/hour. If, initially, tank 1 contains pure water and tank 2 contains 20 kg of salt, set up, but do not solve the initial value problem that describes this situation. Be sure to define your variables and write your final answer in terms of matrices.

**Solution:**

(a) \[
\begin{vmatrix}
1 - \lambda & 2 \\
4 & 2 - \lambda
\end{vmatrix} = (1 - \lambda)(4 - \lambda) - 4 = 0 \implies \lambda(\lambda - 5) = 0 \implies \lambda = 0, 5
\]

\[\lambda = 0: \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} R_2 = -2R_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \implies \mathbf{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}\]

\[\lambda = 5: \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} R_2 = \frac{1}{2}R_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \implies \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\]

The general solution is \( \mathbf{x}(t) = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \)

to which we apply the initial condition to yield \( \mathbf{x}(0) = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \end{bmatrix} \implies \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \end{bmatrix} \)

giving, using Cramer's rule,

\[c_1 = \begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix} = -10 - 2 = -12, \quad c_2 = \begin{vmatrix} -2 & -1 \\ -2 & 1 \end{vmatrix} = -15 - 2 = -17\]

and the solution of the initial value problem as \( \mathbf{x}(t) = 2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + 3e^{5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3e^{5t} - 4 \\ 6e^{5t} + 2 \end{bmatrix} \)

(b) i. \( \text{Tr } A = 2 + k \) and \( |A| = 2k \).
   ii. A. \( k < 0 \implies |A| < 0 \implies \) equilibrium solution/fixed point \((0, 0)\) is a saddle and unstable
   B. \( k = 1 \implies \text{Tr } A = 3, |A| = 2 \) and \( \text{(Tr } A)^2 - 4|A| = 9 - 8 = 1 > 0 \implies \) equilibrium solution/fixed point \((0, 0)\) is an unstable (repelling) node
   iii. \( k > 0 \implies |A| > 0 \) so the only fixed point is \((0, 0)\). Furthermore, \( \text{(Tr } A)^2 - 4|A| = (k - 2)^2 \geq 0 \) for all values of \( k \), implying that the equilibrium/solution/fixed point \((0, 0)\) cannot be a spiral or center since we can only be on or below the parabola in the \( \text{Tr } A-|A| \) parameter plane.

(c) Let \( x_1(t) \) be the amount of salt (kg) in tank 1 and \( x_2(t) \) be the amount of salt (kg) in tank 2. We use the rate of change equals rate in minus rate out.

\[
\frac{dx_1}{dt} = \left(\frac{2 \text{ kg}}{\text{G}}\right)\left(6 \text{ G/hour}\right) + \left(\frac{x_2 \text{ kg}}{100 \text{ G}}\right)\left(1 \text{ G/hour}\right) - \left(\frac{x_1 \text{ kg}}{100 \text{ G}}\right)\left(7 \text{ G/hour}\right) = 12 + \frac{1}{100}x_2 - \frac{7}{100}x_1
\]

\[
\frac{dx_2}{dt} = \left(\frac{x_1 \text{ kg}}{100 \text{ G}}\right)\left(3 \text{ G/hour}\right) - \left(\frac{x_2 \text{ kg}}{100 \text{ G}}\right)\left(3 \text{ G/hour}\right) = \frac{3}{100}x_1 - \frac{3}{100}x_2
\]

\[
\begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} \frac{7}{100} & \frac{3}{100} \\ \frac{3}{100} & -\frac{3}{100} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 12 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 20 \end{bmatrix}
\]
(a) (15 pts) A 40 liter tank is initially half-full of water. A solution containing 10 grams per liter of chemical APPM2360 begins to flow in at 4 liters per minute and the mixed solution flows out at 2 liters per minute. How much APPM2360 is in the tank just before it overflows? Use the integrating factor method in your solution.

Let \( x(t) \) be the amount of chemical APPM2360 at any time \( t \). Then the initial condition is \( x(0) = 0 \) and
\[
\frac{dx}{dt} = \text{rate in} - \text{rate out} = \left(10 \text{ g L}^{-1}\text{min}^{-1}\right) \left(4 \text{ L min}^{-1}\right) - \left(\frac{x}{2t+20} \text{ g L}^{-1}\text{min}^{-1}\right) \left(2 \text{ L min}^{-1}\right) = \frac{dx}{dt} + \frac{x}{t+10} = 40
\]
The integrating factor is \( \mu(t) = t + 10 \) so that
\[
\frac{d}{dt}[(t+10)x] = 40(t+10) \implies \int \frac{d}{dt}[(t+10)x] \, dt = \int 40(t+10) \, dt
\]
\[
(t+10)x = 20(t+10)^2 + C \quad \text{apply initial condition} \quad 0 = 20(10^2) + C \implies C = -2000
\]
\[
x(t) = 20(t+10) - \frac{2000}{t+10}
\]
The tank just begins to overflow when \( t = 10 \) so the amount of chemical in the tank at that time is
\[
x(2) = 20(20) - \frac{2000}{20} = 300 \text{ g}
\]

(b) \( f(t, y) = (y - 1)^{2/3} \) is continuous for all \( t, y \) so Picard’s theorem guarantees the existence of a solution to the differential equation for any initial condition. However \( f_y(t, y) = \frac{2}{3} (y-1)^{-1/3} \) is not continuous if \( y = 1 \). Consequently, Picard’s theorem does not guarantee a unique solution to the initial value problem. There is no violation of the theorem.

(c) Euler’s method is 
\[
y_{n+1} = y_n + hf(t_n, y_n), \text{ } n = 0, 1, 2, \ldots \text{ Thus}
\]
\[
y(1.5) \approx y_1 = y_0 + hf(t_0, y_0) = y(1) + h f(1, 16) = 16 + 0.5(1)(\sqrt{16}) = 18
\]