1. [APPM 2360 Exam (25 pts)] The following problems (a)-(d) are not related.

(a) (11 pts) Consider the linear system \( \mathbf{A} \mathbf{x} = \mathbf{b} \). After several elementary row operations, the augmented matrix of the system is

\[
\begin{bmatrix}
1 & 0 & 1 & | & 1 \\
0 & 0 & 0 & | & 0 \\
0 & 1 & 1 & | & 0
\end{bmatrix}
\]

i. (2 pts) Is the matrix in RREF? If it is not, make it so.
ii. (3 pts) Find a particular solution to the linear system.
iii. (3 pts) Find a basis for and the dimension of the solution space to the associated homogeneous problem.
iv. (3 pts) Use the Nonhomogeneous Principle to write the general solution of the nonhomogeneous system.

(b) (5 pts) If \( \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \), compute \( \mathbf{A}^2 - 5\mathbf{A} - 2\mathbf{I} \).

(c) (5 pts) Is \( \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -5 \\ -4 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^3 \)? Justify your answer.

(d) (4 pts) Let \( \mathcal{W} \) be the set of matrices of the form \( \begin{bmatrix} a & b & c \\ p & q & r \end{bmatrix} \) where \( b, c, p, q, r \) are real numbers and \( a \) is an integer. Is \( \mathcal{W} \) a subspace of \( M_{23} \)? Justify your answer.

2. [APPM 2360 Exam (25 pts)] The following problems (a)-(c) are not related.

(a) (6 pts) Which of the following differential equations have a stable equilibrium solution at \( y(t) = 1 \)? Briefly justify your answer.

i. \( y' = -y \)
ii. \( y' = y(y - 1) \)
iii. \( y' = (y - 1)(y - 2) \)

(b) (12 pts) Consider the system

\[
\begin{align*}
x' &= \left( x - \frac{1}{2} \right) x - xy \\
y' &= (x - 1)y
\end{align*}
\]

i. (6 pts) Find all equilibrium points of the system.
ii. (6 pts) Plot and appropriately label all \( h \) and \( v \) nullclines in the phase plane. At the points \( (1, 1) \) and \( \left( \frac{3}{2}, 1 \right) \) draw arrows indicating the direction that the trajectories through the nullclines would take at those points.

(c) (7 pts) Solve the initial value problem \( y' = \sqrt{t}y \), \( y(1) = 4/9 \). Write your solution as an explicit function of \( t \).

CONTINUED - THERE ARE MORE PROBLEMS AND A TABLE OF LAPLACE TRANSFORMS BELOW
3. [APPM 2360 Exam (40 pts)] The following problems (a)-(d) are not related.

(a) (12 pts) Find the general solution of $y'' - 4y' + 4y = \frac{3e^{2t}}{t}$ on the interval $t > 0$. Look carefully at the equation before deciding how to solve it.

(b) (8 pts) A critically damped mass/spring harmonic oscillator has a restoring constant of 32 N/m and a $\frac{1}{2}$ kg mass. To start the motion, the mass is released from rest 7 units to the left of the equilibrium position.
   
i. (4 pts) Set up, but do not solve, the initial value problem describing the unforced oscillator.
   
ii. (4 pts) Now suppose the aforementioned oscillator is driven with the forcing function in the following graph, where the wavy portion of the graph is a portion of $2 \sin \left(\frac{\pi}{2} t\right)$. Using step functions, how does the differential equation from part (i) change? Do not solve this new equation.

(c) (8 pts) Find $L \{e^{2t} \text{ step}(t-3)\}$.

(d) (12 pts) Solve the initial value problem $y'' + 25y = \delta(t-2)$, $y(0) = 1$, $y'(0) = -5$.

4. [APPM 2360 Exam (35 pts)] The following problems (a)-(c) are not related.

(a) (15 pts) Solve the initial value problem $x' = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} x$, $x(0) = \begin{bmatrix} -1 \\ 8 \end{bmatrix}$. Write your answer as a single vector.

(b) (12 pts) Consider the system $\dot{x} = Ax$ where $A = \begin{bmatrix} 2 & 1 \\ 0 & k \end{bmatrix}$ and $k$ is a parameter.
   
i. (4 pts) Find $\text{Tr} \ A$ and $|A|$.
   
ii. (4 pts) Classify the geometry and stability properties of the system for the following values of $k$.
      
A. $k < 0$     
B. $k = 1$

iii. (4 pts) For $k > 0$ can the fixed point ever be a spiral or center? Briefly explain mathematically why or why not.

(c) (8 pts) Two tanks, each holding 100 gallons (G) of liquid, are interconnected by pipes, with the liquid flowing from tank 1 into tank 2 at a rate of 3 G/hour and from tank 2 into tank 1 at a rate of 1 G/hour. The liquid inside each tank is kept well stirred. A brine solution with a concentration of 2 kg/G of salt flows into tank 1 at a rate of 6 G/hour. The (diluted) solution flows out of the system from tank 1 at 4 G/hour and from tank 2 at 2 G/hour. If, initially, tank 1 contains pure water and tank 2 contains 20 kg of salt, set up, but do not solve the initial value problem that describes this situation. Be sure to define your variables and write your final answer in terms of matrices.

5. [APPM 2360 Exam (25 pts)] The following problems are not related.

(a) (15 pts) A 40 liter tank is initially half-full of water. A solution containing 10 grams per liter of chemical APPM2360 begins to flow in at 4 liters per minute and the mixed solution flows out at 2 liters per minute. How much APPM2360 is in the tank just before it overflows? Use the integrating factor method in your solution.

(b) (5 pts) The initial value problem $dy/dt = (y - 1)^{2/3}$, $y(0) = 1$ has $y = 1$ and $y = 1 + \frac{1}{27} t^3$ as solutions. Does this violate Picard’s theorem? Explain briefly.

(c) (5 pts) Use Euler’s method with a step size of $h = 0.5$ to estimate $y(1.5)$ if $y' = t\sqrt{y}$, $y(1) = 16$.

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Short table of Laplace Transforms: 

$L \{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) \, dt$

In this table, $a, b, c$ are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$
\[
\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}} \quad \mathcal{L}\{e^{at} \cos bt\} = \frac{s-a}{(s-a)^2 + b^2} \quad \mathcal{L}\{e^{at} \sin bt\} = \frac{b}{(s-a)^2 + b^2}
\]

\[
\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n} \quad \mathcal{L}\{e^{at} f(t)\} = F(s-a) \quad \mathcal{L}\{\delta(t-c)\} = e^{-cs}
\]

\[
\mathcal{L}\{t^nf'(t)\} = -F(s) - s \frac{dF(s)}{ds} \quad \mathcal{L}\{f(t-c) \text{step}(t-c)\} = e^{-cs} F(s) \quad \mathcal{L}\{f(t) \text{step}(t-c)\} = e^{-cs} \mathcal{L}\{f(t+c)\}
\]

\[
\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \ldots - f^{(n-1)}(0)
\]