1. [APPM 2360 Exam (18 pts)] Let $L(y) = t^2y'' - ty' + y$ with $t > 1$.

   (a) (6 pts) Show that the set $\{t, t\ln t\}$ forms a basis for the solution space of $L(y) = 0$.

   (b) (12 pts) Find the general solution of $L(y) = \dfrac{t}{\ln t}$. (Hint: u-substitution)

SOLUTION:

(a) Show that both basis vectors are solutions:

$$y_1 = t \implies y_1' = 1 \implies y_1'' = 0 \implies L(y_1) = t^2(0) - t(1) + t = 0$$

$$y_2 = t\ln t \implies y_2\prime = 1 + \ln t \implies y_2'' = \frac{1}{t} \implies L(y_2) = t^2\left(\frac{1}{t}\right) - t\,(1 + \ln t) + t\ln t = 0$$

Show that the set is linearly independent:

$$W[t, t\ln t] = \begin{vmatrix} t & t\ln t \\ 1 & 1 + \ln t \end{vmatrix} = t \neq 0$$

Thus $y_h = c_1t + c_2t\ln t$.

(b) Use variation of parameters with $y_1 = t$ and $y_2 = t\ln t$, choose $y_p = v_1y_1 + v_2y_2$ and write the differential equation as $y'' - \dfrac{1}{t}y' + \dfrac{1}{t^2}y = \dfrac{1}{t\ln t}$ so that $f(t) = \dfrac{1}{t\ln t}$.

$$v_1' = -\frac{t\ln t(1/t\ln t)}{t} = -\frac{1}{t} \implies v_1 = -\int \frac{1}{t}\,\mathrm{d}t = -\ln|t| = -\ln t \;\text{ since } t > 1$$

$$v_2' = \frac{t(1/t\ln t)}{t} = \frac{1}{t\ln t} \implies v_2 = \int \frac{\mathrm{d}t}{t\ln t} \overset{u=\ln t}{=} \int \frac{\mathrm{d}u}{u} = \ln|\ln t| = \ln(\ln t) \;\text{ since } t > 1$$

Thus $y_p = -t\ln t + (t\ln t)\ln(\ln t)$ and $y = c_1t + c_2t\ln t + t\ln t\,[\ln(\ln t) - 1] = c_1t + c_3t\ln t + (t\ln t)\ln(\ln t)$ where $c_3 = c_2 - 1$.

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2. [APPM 2360 Exam (37 pts)] The following problems (a) and (b) are not related.

   (a) (22 pts) Consider the linear operator $L(y) = ay^{(4)} + by''' + cy'' + py' + qy$ where a, b, c, p, q are real constants.

      i. (4 pts) If the characteristic equation of $L(y) = 0$ has the root 1 of algebraic multiplicity 2 and the roots $2 \pm 3i$, write the general solution of $L(y) = 0$.

      ii. (18 pts) For $f(t)$ in each of the following, write the form (**DO NOT SOLVE FOR THE CONSTANTS**) of the particular solution that will be used for the method of undetermined coefficients when solving $L(y) = f(t)$. If the method of undetermined coefficients is not applicable, write N/A.

      A. $f(t) = 2t^{-1}$ B. $f(t) = e^t + \frac{t}{e^t}$ C. $f(t) = te^t + \sin t$

      D. $f(t) = \sin 4t + e^{2t}\cos 3t$ E. $f(t) = t^3e^{t/2}$ F. $f(t) = \tan 3t + 2$

   (b) (15 pts) Use the method of undetermined coefficients to solve the initial value problem

$$y''' - 2y'' = 4,\; y(0) = 9,\; y'(0) = 1,\; y''(0) = 14$$

SOLUTION:

(a) i. $y(t) = c_1e^t + c_2te^t + e^{2t}(c_3\cos 3t + c_4\sin 3t)$

ii. A. N/A

B. $y_p = At^2e^t + (Bt + C)e^{-t}$

C. $y_p = (At^3 + Bt^2)e^t + C\sin t + D\cos t$

D. $y_p = A\sin 4t + B\cos 4t + te^{2t}\,(C\sin 3t + D\cos 3t)$

E. $y_p = (At^3 + Bt^2 + Ct + D)e^{t/2}$

F. N/A

(b) The characteristic equation of the homogeneous equation is $r^2(r-2)=0$ giving $y_h = c_1 + c_2 t + c_3 e^{2t}$. Substituting $y_p = At^2$ into the nonhomogeneous ODE yields $0 - 2(2A) = 4 \implies A = -1$ so that $y = y_h + y_p = c_1 + c_2 t + c_3 e^{2t} - t^2$. Now apply the initial conditions:

$$y(0) = c_1 + c_3 = 9$$
$$y'(t) = c_2 + 2c_3 e^{2t} - 2t \implies y'(0) = c_2 + 2c_3 = 1$$
$$y''(t) = 4c_3 e^{2t} - 2 \implies y''(0) = 4c_3 - 2 = 14 \implies c_3 = 4$$
$$c_2 + 8 = 1 \implies c_2 = -7 \text{ and } c_1 + 4 = 9 \implies c_1 = 5$$

Thus, $y = 5 - 7t - t^2 + 4e^{2t}$.

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3. [APPM 2360 Exam (24 pts)] The following problems (a) and (b) are not related.

(a) (3 pts) Find the value(s) of b and m for which the harmonic oscillator governed by $m\ddot{x} + b\dot{x} + x = \cos(2\pi t)$ will have unbounded solutions.

(b) (21 pts) A 3 kg mass is attached to a spring which in turn is attached to a wall. The entire apparatus is on a horizontal shelf and immersed in a fluid that offers a damping force equal to 6 times the instantaneous velocity. The oscillator is unforced and has a Spring-o-Matic dial marked with values of k that allows you to adjust the spring (restoring) constant, k.

i. (6 pts) As the Spring-o-Matic dial is moved from $k = 5$ to 3 to 1, identify the type of damping (critical, over, under) that occurs for each of these values of k.

ii. (15 pts) If $k = 3$ and the mass is initially pulled 2 m to the right of the equilibrium and then imparted a 4 m/sec velocity to the left, will it pass through the equilibrium? If so, when? If not, briefly explain why not.

SOLUTION:

(a) The frequency of the forcing function is $\omega_f = 2\pi$. The circular frequency of the oscillator is $\omega_0 = \sqrt{1/m}$. For solutions to be unbounded the oscillator needs to be undamped with $\omega_f = \omega_0$ (resonance). These conditions require, respectively, $b = 0$ and $2\pi = \sqrt{1/m} \implies m = 1/(4\pi^2)$.

(b) i. If $x(t)$ is the displacement of the oscillator from the equilibrium position, the equation describing the oscillator is $3\ddot{x} + 6\dot{x} + kx = 0$. The type of damping is determined by the quantity $b^2 - 4mk = 36 - 4(3)k = 12(3-k)$. If $k = 5$ this quantity is negative implying underdamping, $k = 3$ makes this quantity vanish, implying critical damping, while $k = 1$ makes the quantity positive, implying overdamping.

ii. If $k = 3$ we have $3\ddot{x} + 6\dot{x} + 3x = 0$, $x(0) = 2$, $\dot{x}(0) = -4$ with characteristic equation

$$3r^2 + 6r + 3 = 3(r^2 + 2r + 1) = 0 \implies r = -1 \text{ with multiplicity } 2$$

so that the general solution to the ODE is $x(t) = c_1 e^{-t} + c_2 t e^{-t}$. Applying the initial condition gives

$$x(0) = c_1 + 0 = 2 \implies c_1 = 2$$
$$\dot{x}(t) = -2e^{-t} - c_2 t e^{-t} + c_2 e^{-t} \implies \dot{x}(0) = -2 + c_2 = -4 \implies c_2 = -2$$
$$x(t) = 2e^{-t}(1-t)$$

Passing through the equilibrium means that $x = 0$ which will occur when $t = 1$ sec.

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4. [APPM 2360 Exam (21 pts)] The following problems are not related.

(a) (6 pts) Convert the initial value problem $3y''' + 9ty'' - 24\sqrt{t}y' - 30y = 60t - 36$, $y(1) = 4$, $y'(1) = -3$, $y''(1) = 0$ into a system of first order equations with appropriate initial condition. If possible, write this system as a matrix equation. If not possible, explain why not.

(b) (15 pts) Use Laplace transform techniques to solve the initial value problem $y'' - 4y' + 20y = 40$, $y(0) = y'(0) = 0$.

SOLUTION:

(a) Rewrite the differential equation as $y''' = -3ty'' + 8\sqrt{t}y' + 10y + 20t - 12$ and let $x_1 = y$, $x_2 = y'$, $x_3 = y''$. Then

$$\begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = x_3 \\ x_3' &= y''' = -3ty'' + 8\sqrt{t}y' + 10y + 20t - 12 \\ &= 10x_1 + 8\sqrt{t}x_2 - 3tx_3 + 20t - 12 \end{aligned}$$

Since the original equation is linear we can write this as a matrix equation

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 10 & 8\sqrt{t} & -3t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 20t - 12 \end{bmatrix} \qquad \begin{bmatrix} x_1(1) \\ x_2(1) \\ x_3(1) \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}$$

(b)

$$\begin{aligned} &\mathscr{L}\left\{y'' - 4y' + 20y = 40\right\} \\ &s^2Y(s) - sy(0) - y'(0) - 4\left[sY(s) - y(0)\right] + 20Y(s) = \frac{40}{s} \\ &Y(s) = \frac{40}{s(s^2 - 4s + 20)} \qquad \text{partial fractions} \\ &\frac{40}{s(s^2 - 4s + 20)} = \frac{A}{s} + \frac{Bs + C}{s^2 - 4s + 20} \\ &40 = A(s^2 - 4s + 20) + (Bs + C)s \end{aligned}$$

$$\begin{aligned} s = 0: &\quad 40 = 20A \implies A = 2 \\ s = 1: &\quad 40 = 2(17) + B + C \implies B + C = 6 \\ s = -1: &\quad 40 = 2(25) + (-B + C)(-1) \implies B - C = -10 \\ &B = -2 \quad C = 8 \end{aligned}$$

So

$$\begin{aligned} Y(s) &= \frac{2}{s} + \frac{-2s + 8}{s^2 - 4s + 20} = \frac{2}{s} - 2\left[\frac{s - 4}{s^2 - 4s + 20}\right] = \frac{2}{s} - 2\left[\frac{s - 2 - 2}{(s - 2)^2 + 16}\right] \\ &= \frac{2}{s} - 2\left[\frac{s - 2}{(s - 2)^2 + 4^2}\right] + \frac{4}{(s - 2)^2 + 4^2} \end{aligned}$$

Taking inverse Laplace Transforms gives

$$y(t) = \mathscr{L}^{-1}\left\{Y(s)\right\} = \mathscr{L}^{-1}\left\{\frac{2}{s} - 2\left[\frac{s - 2}{(s - 2)^2 + 4^2}\right] + \frac{4}{(s - 2)^2 + 4^2}\right\}$$

$$y(t) = 2 + e^{2t}\left(\sin 4t - 2\cos 4t\right)$$

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