1. [APPM 2360 Exam (24 pts)] Consider the following matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Compute the following, if possible. If not possible, write "NOT DEFINED" and provide a brief explanation why.

(a) $\mathbf{AB}$ (b) $\mathbf{AC}$ (c) $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ (d) $\left|\mathbf{C}^{-1}\right|$ (e) $\mathbf{B}^{\mathrm{T}}\mathbf{B}$ (f) $\mathbf{C}-\mathbf{C}^{\mathrm{T}}$ (g) $\left(\mathbf{BB}^{\mathrm{T}}\right)^{-1}$ (h) $\mathbf{CA}^{-1}$

SOLUTION:

(a)

$$\mathbf{AB} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

(b) NOT DEFINED - number of columns in $\mathbf{A}$ is not equal to the number of rows in $\mathbf{C}$

(c)

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 5 \\ 2 & 1 & 3 \\ 5 & 3 & 10 \end{bmatrix}$$

(d) NOT DEFINED - $\mathbf{C}$ is singular (noninvertible)

(e)

$$\mathbf{B}^{\mathrm{T}}\mathbf{B} = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix}$$

(f)

$$\mathbf{C}-\mathbf{C}^{\mathrm{T}} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(g) NOT DEFINED

$$\mathbf{BB}^{\mathrm{T}} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

is singular (noninvertible)

(h) $\mathbf{A}$ is not square and thus does not possess an inverse

■

2. [APPM 2360 (32 pts)] The following problems are not related. Be sure to provide thorough justification for your answers.

(a) (8 pts) For what value(s) of t is the set of vectors $\left\{ \begin{bmatrix} t \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 4t \end{bmatrix} \right\}$ linearly independent?

(b) (8 pts) Does the set of vectors $\left\{2-x, 2x-x^2, 6-5x+x^2, x\right\}$ form a basis for $\mathbb{P}_2$?

(c) (8 pts) Are the functions $\{t, \sin t, \cos t\}$ linearly independent on the real line?

(d) (8 pts) Consider the vector space $\mathbb{M}_{22}$, the set of all 2×2 matrices with the standard operations for matrix addition and scalar multiplication.

i. Let $\mathbb{W}_1$ be the set of matrices of the form $\begin{bmatrix} a & b \\ -b & c \end{bmatrix}$ where a, b, c are real numbers. Is $\mathbb{W}_1$ a subspace of $\mathbb{M}_{22}$?

ii. Let $\mathbb{W}_2$ be the set of matrices of the form $\begin{bmatrix} a & 2 \\ -2 & b \end{bmatrix}$ where a, b are real numbers. Is $\mathbb{W}_2$ a subspace of $\mathbb{M}_{22}$?

SOLUTION:

APPM 2360 Exam APPM 2360 Exam APPM 2360 Exam

(a) To be linearly independent, the only solution to

$$c_1\begin{bmatrix}t\\2\\1\end{bmatrix}+c_2\begin{bmatrix}1\\0\\1\end{bmatrix}+c_3\begin{bmatrix}4\\1\\4t\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

must be $c_1 = c_2 = c_3 = 0$. This is equivalent to requiring

$$\begin{bmatrix}t&1&4\\2&0&1\\1&1&4t\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

to have only the trivial solution which will be the case if the determinant of the coefficient matrix is nonzero.

$$\begin{vmatrix}t&1&4\\2&0&1\\1&1&4t\end{vmatrix}=9-9t=0 \implies t=1$$

Therefore, the vectors will be linearly independent for all real $t \neq 1$.

(b) No, the dimension of $\mathbb{P}_2$ is 3 and the set contains 4 vectors.

(c) Compute the Wronskian:

$$W(t)=\begin{vmatrix}t&\sin t&\cos t\\1&\cos t&-\sin t\\0&-\sin t&-\cos t\end{vmatrix}=t(-1)^{1+1}\begin{vmatrix}\cos t&-\sin t\\-\sin t&-\cos t\end{vmatrix}+(1)(-1)^{2+1}\begin{vmatrix}\sin t&\cos t\\-\sin t&-\cos t\end{vmatrix}=-t$$

Since this is not zero for at least one point in $\mathbb{R}$, the functions are linearly independent on the real line.

(d) i. Let $\vec{\mathbf{u}}=\begin{bmatrix}u_1&u_2\\-u_2&u_3\end{bmatrix}$ and $\vec{\mathbf{v}}=\begin{bmatrix}v_1&v_2\\-v_2&v_3\end{bmatrix}$ be two vectors in $\mathbb{W}_1$ and A, B be real numbers. Then

$$A\vec{\mathbf{u}}+B\vec{\mathbf{v}}=A\begin{bmatrix}u_1&u_2\\-u_2&u_3\end{bmatrix}+B\begin{bmatrix}v_1&v_2\\-v_2&v_3\end{bmatrix}=\begin{bmatrix}Au_1&Au_2\\-Au_2&Au_3\end{bmatrix}+\begin{bmatrix}Bv_1&Bv_2\\-Bv_2&Bv_3\end{bmatrix}=\begin{bmatrix}Au_1+Bv_1&Au_2+Bv_2\\-Au_2-Bv_2&Au_3+Bv_3\end{bmatrix}$$

Since the off-diagonal elements sum to 0, $A\vec{\mathbf{u}}+B\vec{\mathbf{v}} \in \mathbb{W}_1$, implying that $\mathbb{W}_1$ is closed under vector addition and scalar multiplication and is thus a subspace of $\mathbb{M}_{22}$.

ii. Since the zero vector $\begin{bmatrix}0&0\\0&0\end{bmatrix} \notin \mathbb{W}_2$, $\mathbb{W}_2$ is not a subspace of $\mathbb{M}_{22}$.

3. [APPM 2360 Exam (20 pts)] The following problems are not related.

(a) (5 pts) Is $\vec{\mathbf{v}}=\begin{bmatrix}2\\-4\\6\end{bmatrix}$ an eigenvector of $\mathbf{A}=\begin{bmatrix}-1&-1&1\\-2&0&-2\\3&-3&1\end{bmatrix}$? Hint: Use an appropriate definition.

(b) (5 pts) Find the eigenvalues of $\mathbf{A}=\begin{bmatrix}3&5\\-1&-1\end{bmatrix}$. Simplify your answer and **DO NOT** find the eigenvectors.

(c) (10 pts) The characteristic equation for the matrix $\mathbf{A}=\begin{bmatrix}1&2&-2\\-2&5&-2\\-6&6&-3\end{bmatrix}$ is $(\lambda-3)^2(\lambda+3)=0$. Find a basis for and the dimension of the eigenspace corresponding to the repeated eigenvalue.

SOLUTION:

(a)

$$\mathbf{A}\vec{\mathbf{v}}=\begin{bmatrix}-1&-1&1\\-2&0&-2\\3&-3&1\end{bmatrix}\begin{bmatrix}2\\-4\\6\end{bmatrix}=\begin{bmatrix}8\\-16\\24\end{bmatrix}=4\begin{bmatrix}2\\-4\\6\end{bmatrix}$$

Since $\mathbf{A}\vec{\mathbf{v}}=4\vec{\mathbf{v}}$, $\vec{\mathbf{v}}$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda = 4$.

(b)

$$\begin{vmatrix}3-\lambda&5\\-1&-1-\lambda\end{vmatrix}=(3-\lambda)(-1-\lambda)+5=-3-3\lambda+\lambda+\lambda^2+5=\lambda^2-2\lambda+2=0$$

$$\lambda=\frac{2\pm\sqrt{(-2)^2-4(1)(2)}}{2}=\frac{2\pm\sqrt{-4}}{2}=1\pm i$$

APPM 2360 Exam

(c) We need to solve the system $(\mathbf{A} - 3\mathbf{I})\vec{\mathbf{v}} = \vec{\mathbf{0}}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 2 & -2 & 0 \\ -2 & 2 & -2 & 0 \\ -6 & 6 & -6 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies v_1 = v_2 - v_3$$

v_2 and v_3 are free variables so that

$$\vec{\mathbf{v}} = \begin{bmatrix} s-t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

so a basis for the eigenspace $\mathbb{E}_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ which has dimension 2.

4. [APPM 2360 Exam (24 pts)] The following problems are not related.

(a) (4 pts) For what value(s) of k can Cramer's Rule be used to solve the linear system

$$x + ky = 10$$
$$7x + 21y = -4$$

(b) (8 pts) Use Gauss-Jordan elimination to find the inverse matrix and use the inverse to solve the system

$$x_1 + 2x_2 = 6$$
$$3x_1 + 4x_2 = 14$$

No points for using a formula to find the inverse matrix.

(c) (12 pts) Consider the linear system

$$\begin{aligned} x_1 - x_2 - x_3 &= -1 \\ 2x_1 + x_2 - 2x_3 &= -2 \\ -x_1 + 2x_2 + x_3 &= 1 \end{aligned}$$

i. (8 pts) Use Gauss-Jordan elimination to transform the augmented matrix of the system to RREF.

ii. (3 pts) Use the Nonhomogeneous Principle to write the solution of the linear system in the form $\vec{\mathbf{x}} = \vec{\mathbf{x}}_h + \vec{\mathbf{x}}_p$.

iii. (1 pts) What is the dimension of the solution space of the associated homogeneous system?

SOLUTION:

(a) The coefficient matrix of the system, $\begin{bmatrix} 1 & k \\ 7 & 21 \end{bmatrix}$, must have nonzero determinant to use Cramer's Rule.

$$\begin{vmatrix} 1 & k \\ 7 & 21 \end{vmatrix} = 21 - 7k = 0 \implies k = 3$$

If $k \neq 3$, then Cramer's Rule can be used to solve the linear system.

(b)

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \xrightarrow{R_2^* = -3R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \xrightarrow{R_1^* = R_2 + R_1} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right] \xrightarrow{R_2^* = -\frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right]$$

Thus $\mathbf{A}^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$ and

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 6 \\ 14 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

(c) i.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 2 & 1 & -2 & -2 \\ -1 & 2 & 1 & 1 \end{array}\right] \begin{array}{c} R_2^* = -2R_1 + R_2 \\ R_3^* = R_1 + R_3 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \begin{array}{c} R_1^* = R_3 + R_1 \\ R_2^* = -3R_3 + R_2 \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] R_2 \leftrightarrow R_3 \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

APPM 2360 Exam APPM 2360 Exam APPM 2360 Exam

This then gives $x_1 = -1 + x_3$ and $x_2 = 0$ with x_3 as free variable so that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1+t \\ 0 \\ t \end{bmatrix}$$

ii. A particular solution is obtained by letting $t = 0$ or $\vec{\mathbf{x}}_p = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$. The RREF of the augmented matrix corresponding to the associated homogeneous system is

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

implying that $x_1 = x_3, x_2 = 0$ with $x_3 = t$ a free variable. Therefore, $\vec{\mathbf{x}}_h = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. The Nonhomogeneous Principle then gives the solution to the linear system as

$$\vec{\mathbf{x}} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}$$

iii. The dimension of the solution space of the associated homogeneous system is 1.

■

APPM 2360 Exam APPM 2360 Exam APPM 2360 Exam