1. [APPM 2360 Exam (20 pts)] The following problems are not related.

(a) (6 pts) What conclusions can be drawn from Picard’s Theorem regarding the existence and uniqueness of solutions to the initial value problem \( y' = t/y, \ y(2) = 0 \)? Briefly explain.

(b) (6 pts) With a step size of \( h = 0.5 \), use Euler’s method to approximate the solution of the IVP \( y' = t - y, \ y(1) = 2 \) at \( t = 2 \).

(c) (8 pts) Consider the following system of differential equations

\[
\begin{align*}
x' &= 1 + x - y \\
y' &= -1 + x^2 + y^2
\end{align*}
\]

i. (4 pts) Sketch and label the \( h \) and \( v \) nullclines in the phase plane.

ii. (4 pts) Find all equilibrium points, if any exist.

**SOLUTION:**

(a) We have \( f(t, y) = t/y \), which is continuous if \( y \neq 0 \) and \( f_y = -t/y^2 \) which is continuous if \( y \neq 0 \). \( f(2, 0) \) is not defined and thus \( f(t, y) \) is not continuous at \((2, 0)\). Similarly for \( f_y(2, 0) \). We can draw no conclusions from Picard’s theorem regarding the existence or uniqueness of solutions to the IVP.

(b) Using \( y_{n+1} = y_n + h(t_n - y_n) \), \( n = 0, 1 \) we have

\[
\begin{align*}
y(1.5) &\approx y_1 = y_0 + \frac{1}{2} (5 - y_0) = 2 + \frac{1}{2} (1 - 2) = \frac{3}{2} \\
y(2.0) &\approx y_2 = y_1 + \frac{1}{2} (t_1 - y_1) = \frac{3}{2} + \frac{1}{2} \left( \frac{1}{2} - \frac{3}{2} \right) = \frac{1}{2}
\end{align*}
\]

(c) i. The \( v \) nullclines occur where \( x' = 1 + x - y = 0 \implies y = x + 1 \) is the only \( v \) nullcline (dashed in figure). The \( h \) nullclines occur where \( y' = -1 + x^2 + y^2 = 0 \implies x^2 + y^2 = 1 \) is the only \( h \) nullcline (solid in figure).

![nullclines](image)

ii. Equilibrium points occur where \( x' = 0 \) and \( y' = 0 \) simultaneously. Using \( y = x + 1 \) in the equation for the circle gives

\[
x^2 + (1 + x)^2 = 1 \implies 2x(x + 1) = 0 \implies x = 0, x = -1 \implies y = 1, y = 0 \text{ respectively. Equilibrium points are } (-1, 0) \text{ and } (0, 1). \text{ These are also evident from the sketch (intersection of } h \text{ nullclines and } v \text{ nullclines).}
\]

2. [APPM 2360 (18 pts)] The following problems are not related.

(a) (10 pts) Consider the differential equation \( y' = y^2(y + 4)(y^2 - 4) \).

i. (4 pts) Find all equilibrium solutions and their stability.

ii. (6 pts) Plot the phase line for the differential equation.

(b) (8 pts) Given the differential equation \( y' - y + 2t = 0 \), draw the isoclines corresponding to slopes of 1, 0, -1. Be sure to include the line segments showing the slope on each isocline.

**SOLUTION:**

(a) We can write the ODE as \( y' = y^2(y+4)^2(y-2)(y+2) \) showing that the equilibrium solutions are \( y = -4, y = -2, y = 0, y = 2 \).

i. \[
\begin{align*}
y > 2 : & \quad y' > 0 \\
0 < y < 2 : & \quad y' < 0 \\
-2 < y < 0 : & \quad y' < 0 \\
-4 < y < -2 : & \quad y' > 0 \\
y < -4 : & \quad y' > 0
\end{align*}
\]

Thus \( y = 0 \) and \( y = -4 \) are semistable, \( y = 2 \) is unstable and \( y = -2 \) is stable.
ii. Phase line.

(b) Isoclines are the lines $y = 2t + k$ with $k = -1, 0, 1$.

3. [APPM 2360 Exam (22 pts)] Consider the initial value problem $y' = \frac{y^2 + 2ty}{t^2}$, $y(1) = -2$, $t > 0$.

(a) (2 pts) Letting $v = y/t$, show that $y' = tv' + v$.

(b) (2 pts) Use part (a) to show that the original differential equation can be rewritten as $tv' = v^2 + v$.

(c) (12 pts) Solve the differential equation in part (b).

(d) (3 pts) Find the general solution to the original differential equation, writing your answer explicitly as $y(t) = \ldots$.

(e) (3 pts) Find the solution to the original initial value problem.

**SOLUTION:**

(a) 

$v = y/t \implies v' = \frac{ty' - y}{t^2} \implies tv' = y' - \frac{y}{t} \implies y' = tv' + v$

(b) The differential equation can be rewritten as $y' = \left(\frac{y}{t}\right)^2 + 2\left(\frac{y}{t}\right)$ so that using part (a) we have

$tv' + v = v^2 + 2v \implies tv' = v^2 + v$

(c) The DE in part (b) is separable.

$t \frac{dv}{dt} = v^2 + v \implies \frac{dv}{v^2 + v} = \frac{dt}{t}$
The left hand side is integrated using partial fractions:

\[
\int \frac{dv}{v(v+1)} = \int \frac{dt}{t} \\
\int \frac{dv}{v} - \int \frac{dv}{v+1} = \int \frac{dt}{t} \\
\ln \left| \frac{v}{v+1} \right| = \ln |t| + \widetilde{C} \\
\frac{v}{v+1} = Ct \\
v = \frac{Ct}{1-Ct}
\]

(d) Backing out the original substitution yields

\[
y = \frac{Ct}{1-Ct} \implies y(t) = \frac{Ct^2}{1-Ct}
\]

(e) Applying the initial condition gives

\[-2 = \frac{C(1^2)}{1-C(1)} \implies C = 2 \implies y(t) = \frac{2t^2}{1-2t}
\]

4. [APPM 2360 Exam (16 pts)] Consider the differential equation \(ty' - 2y = t^4e^{t^2}, t > 0\).

(a) (2 pts) Show that \(y_h(t) = Ct^2\) is a solution of the associated homogeneous equation. \(C\) is an arbitrary constant.

(b) (12 pts) Use the Euler-Lagrange two-stage method (variation of parameters) to find a particular solution to the nonhomogeneous equation. Show all your work.

(c) (2 pts) Use the Nonhomogeneous Principle to write the general solution to the nonhomogeneous differential equation.

SOLUTION:

(a) We have \(ty_h - 2y_h = t[C(2t)] - 2(Ct^2) = 2Ct^2 - 2Ct^2 = 0\).

(b) Let \(y_p = v(t)t^2\) so that \(y'_p = 2tv(t) + v'(t)t^2\). Substituting into the nonhomogeneous DE yields

\[
y'_p - 2y_p = tv(t) + v'(t)t^2 - 2v(t)t^2 = t^3v'(t) = t^4e^{t^2} \\
v' = te^{t^2} \\
v(t) = \int te^{t^2} dt = \frac{1}{2}e^{t^2} \\
y_p = \frac{1}{2}t^2e^{t^2}
\]

(c)

\[y(t) = y_h(t) + y_p(t) = Ct^2 + \frac{1}{2}t^2e^{t^2}\]

5. [APPM 2360 Exam (24 pts)] The following problems are not related.

(a) (10 pts) Suppose that a room containing 1000 ft\(^3\) of air is originally free of a certain pollutant. Beginning at time \(t = 0\) air, containing the pollutant at a concentration of 0.05 g/100 ft\(^3\), is introduced into the room at a rate of 0.1 ft\(^3\)/min, and the well-circulated mixture is allowed to leave the room at the same rate. Let \(x(t)\) be the amount of pollutant (grams) at time \(t\). Set up, but DO NOT SOLVE, the initial value problem governing this situation. Simplify your answer.

(b) (14 pts) Consider the differential equation \(xy' + 2y = -\frac{\sin x}{x}, x > 0\).

i. (12 pts) Using the integrating factor method, find the solution that passes through the point \((\pi/2, 0)\).

ii. (2 pts) For what value of \(x\) does \(y = 1/x^2\)?

SOLUTION:
(a) \[
\frac{dx}{dt} = \text{rate in} - \text{rate out} = \left( \frac{0.05 \text{ g}}{100 \text{ ft}^3} \right) \left( \frac{0.1 \text{ ft}^3}{\text{min}} \right) - \left( \frac{x \text{ g}}{1000 \text{ ft}^3} \right) \left( \frac{0.1 \text{ ft}^3}{\text{min}} \right)
\]
\[
\frac{dx}{dt} + \frac{x}{10000} = \frac{1}{20000}, \quad x(0) = 0
\]

(b) i. Rewrite the equation as \( y' + \frac{2}{x} y = -\frac{\sin(x)}{x^2} \). With \( p(x) = \frac{2}{x} \), the integrating factor is
\[
\int \frac{2}{x} \, dx = 2 \ln|x| = \ln x^2 \implies \mu(x) = x^2
\]
Multiplying by the integrating factor yields
\[
(x^2 y)' = -\sin x \implies x^2 y = \cos x + C \implies y(x) = \frac{\cos x}{x^2} + \frac{C}{x^2}
\]
Applying the initial condition \( y(\pi/2) = 0 \) gives
\[
y(\pi/2) = 0 = 0 + C/(\pi^2/4) \implies C = 0
\]
so the solution is \( y(x) = \frac{\cos x}{x^2} \).

ii. We need to determine when \( y(x) = \frac{\cos x}{x^2} = -\frac{1}{\pi^2} \). By inspection, this occurs when \( x = \pi \).