1. [APPM 2360 Exam (20 pts)] The following problems are not related.

   (a) (6 pts) What conclusions can be drawn from Picard's Theorem regarding the existence and uniqueness of solutions to the initial value problem $y' = t/y,\ y(2) = 0$? Briefly explain.

   (b) (6 pts) With a step size of $h = 0.5$, use Euler's method to approximate the solution of the IVP $y' = t - y,\ y(1) = 2$ at $t = 2$.

   (c) (8 pts) Consider the following system of differential equations

$$
\begin{aligned}
x' &= 1 + x - y \\
y' &= -1 + x^2 + y^2
\end{aligned}
$$

   i. (4 pts) Sketch and label the h and v nullclines in the phase plane.

   ii. (4 pts) Find all equilibrium points, if any exist.

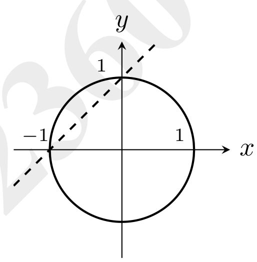
SOLUTION:

(a) We have $f(t, y) = t/y$, which is continuous if $y \neq 0$ and $f_y = -t/y^2$ which is continuous if $y \neq 0$. $f(2, 0)$ is not defined and thus $f(t, y)$ is not continuous at $(2, 0)$. Similarly for $f_y(2, 0)$. We can draw no conclusions from Picard's theorem regarding the existence or uniqueness of solutions to the IVP.

(b) Using $y_{n+1} = y_n + h(t_n - y_n),\ n = 0, 1$ we have

$$
\begin{aligned}
y(1.5) &\approx y_1 = y_0 + \tfrac{1}{2}(t_0 - y_0) = 2 + \tfrac{1}{2}(1 - 2) = \tfrac{3}{2} \\
y(2.0) &\approx y_2 = y_1 + \tfrac{1}{2}(t_1 - y_1) = \tfrac{3}{2} + \tfrac{1}{2}\left(\tfrac{3}{2} - \tfrac{3}{2}\right) = \tfrac{3}{2}
\end{aligned}
$$

(c) i. The v nullclines occur where $x' = 1 + x - y = 0 \implies y = x + 1$ is the only v nullcline (dashed in figure). The h nullclines occur where $y' = -1 + x^2 + y^2 = 0 \implies x^2 + y^2 = 1$ is the only h nullcline (solid in figure).



ii. Equilibrium points occur where $x' = 0$ and $y' = 0$ simultaneously. Using $y = x + 1$ in the equation for the circle gives $x^2 + (1 + x)^2 = 1 \implies 2x(x + 1) = 0 \implies x = 0, x = -1 \implies y = 1, y = 0$ respectively. Equilibrium points are $(-1, 0)$ and $(0, 1)$. These are also evident from the sketch (intersection of h nullclines and v nullclines).

2. [APPM 2360 (18 pts)] The following problems are not related.

   (a) (10 pts) Consider the differential equation $y' = y^2(y + 4)^2(y^2 - 4)$.

   i. (4 pts) Find all equilibrium solutions and their stability.

   ii. (6 pts) Plot the phase line for the differential equation.

   (b) (8 pts) Given the differential equation $y' - y + 2t = 0$, draw the isoclines corresponding to slopes of $1, 0, -1$. Be sure to include the line segments showing the slope on each isocline.

SOLUTION:

(a) We can write the ODE as $y' = y^2(y+4)^2(y-2)(y+2)$ showing that the equilibrium solutions are $y = -4, y = -2, y = 0, y = 2$.

i.

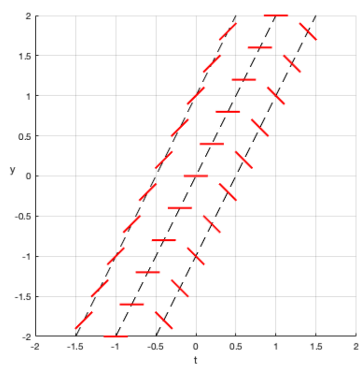
$$
\begin{aligned}
y > 2 :&\quad y' > 0 \\
0 < y < 2 :&\quad y' < 0 \\
-2 < y < 0 :&\quad y' < 0 \\
-4 < y < -2 :&\quad y' > 0 \\
y < -4 :&\quad y' > 0
\end{aligned}
$$

Thus $y = 0$ and $y = -4$ are semistable, $y = 2$ is unstable and $y = -2$ is stable.

ii. Phase line.

$$\uparrow$$
$$2 \ \bigcirc$$
$$\downarrow$$
$$0 \ ◓$$
$$\downarrow$$
$$-2 \ \bullet$$
$$\uparrow$$
$$-4 \ ◒$$
$$\uparrow$$

(b) Isoclines are the lines $y = 2t + k$ with $k = -1, 0, 1$.



3. [APPM 2360 Exam (22 pts)] Consider the initial value problem $y' = \dfrac{y^2 + 2ty}{t^2}$, $y(1) = -2$, $t > 0$.

(a) (2 pts) Letting $v = y/t$, show that $y' = tv' + v$.

(b) (2 pts) Use part (a) to show that the original differential equation can be rewritten as $tv' = v^2 + v$.

(c) (12 pts) Solve the differential equation in part (b).

(d) (3 pts) Find the general solution to the original differential equation, writing your answer explicitly as $y(t) = \cdots$.

(e) (3 pts) Find the solution to the original initial value problem.

SOLUTION:

(a)

$$v = y/t \implies v' = \frac{ty' - y}{t^2} \implies tv' = y' - \frac{y}{t} \implies y' = tv' + v$$

(b) The differential equation can be rewritten as $y' = \left(\frac{y}{t}\right)^2 + 2\left(\frac{y}{t}\right)$ so that using part (a) we have

$$tv' + v = v^2 + 2v \implies tv' = v^2 + v$$

(c) The DE in part (b) is separable.

$$t\frac{\mathrm{d}v}{\mathrm{d}t} = v^2 + v \implies \frac{\mathrm{d}v}{v^2 + v} = \frac{\mathrm{d}t}{t}$$

The left hand side is integrated using partial fractions:

$$\int \frac{\mathrm{d}v}{v(v+1)} = \int \frac{\mathrm{d}t}{t}$$
$$\int \frac{\mathrm{d}v}{v} - \int \frac{\mathrm{d}v}{v+1} = \int \frac{\mathrm{d}t}{t}$$
$$\ln\left|\frac{v}{v+1}\right| = \ln|t| + \widetilde{C}$$
$$\frac{v}{v+1} = Ct$$
$$v = \frac{Ct}{1-Ct}$$

(d) Backing out the original substitution yields

$$\frac{y}{t} = \frac{Ct}{1-Ct} \implies y(t) = \frac{Ct^2}{1-Ct}$$

(e) Applying the initial condition gives

$$-2 = \frac{C(1^2)}{1-C(1)} \implies C = 2 \implies y(t) = \frac{2t^2}{1-2t}$$

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4. [APPM 2360 Exam (16 pts)] Consider the differential equation $ty' - 2y = t^4 e^{t^2},\ t > 0$.

(a) (2 pts) Show that $y_h(t) = Ct^2$ is a solution of the associated homogeneous equation. C is an arbitrary constant.

(b) (12 pts) Use the Euler-Lagrange two-stage method (variation of parameters) to find a particular solution to the nonhomogeneous equation. Show all your work.

(c) (2 pts) Use the Nonhomogeneous Principle to write the general solution to the nonhomogeneous differential equation.

SOLUTION:

(a) We have $ty_h - 2y_h = t[C(2t)] - 2(Ct^2) = 2Ct^2 - 2Ct^2 = 0$.

(b) Let $y_p = v(t)t^2$ so that $y_p' = 2tv(t) + v'(t)t^2$. Substituting into the nonhomogeneous DE yields

$$ty_p' - 2y_p = t\left[2tv(t) + v'(t)t^2\right] - 2v(t)t^2 = t^3 v'(t) = t^4 e^{t^2}$$
$$v'(t) = te^{t^2}$$
$$v(t) = \int te^{t^2}\,\mathrm{d}t \overset{u=t^2}{=} \frac{1}{2}e^{t^2}$$
$$y_p = \frac{1}{2}t^2 e^{t^2}$$

(c)

$$y(t) = y_h(t) + y_p(t) = Ct^2 + \frac{1}{2}t^2 e^{t^2}$$

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5. [APPM 2360 Exam (24 pts)] The following problems are not related.

(a) (10 pts) Suppose that a room containing 1000 ft^3 of air is originally free of a certain pollutant. Beginning at time $t = 0$ air, containing the pollutant at a concentration of 0.05 g/100 ft^3, is introduced into the room at a rate of 0.1 ft^3/min, and the well-circulated mixture is allowed to leave the room at the same rate. Let $x(t)$ be the amount of pollutant (grams) at time t. Set up, but **DO NOT SOLVE**, the initial value problem governing this situation. Simplify your answer.

(b) (14 pts) Consider the differential equation $xy' + 2y = -\dfrac{\sin x}{x},\ x > 0$.

i. (12 pts) Using the integrating factor method, find the solution that passes through the point $(\pi/2, 0)$.

ii. (2 pts) For what value of x does $y = -1/\pi^2$?

SOLUTION:

(a)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \text{rate in} - \text{rate out} = \left(\frac{0.05 \text{ g}}{100 \text{ ft}^3}\right)\left(\frac{0.1\text{ft}^3}{\text{min}}\right) - \left(\frac{x \text{ g}}{1000 \text{ ft}^3}\right)\left(\frac{0.1\text{ft}^3}{\text{min}}\right)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{x}{10000} = \frac{1}{20000},\ x(0) = 0$$

(b) i. Rewrite the equation as $y' + \frac{2}{x}y = -\frac{\sin x}{x^2}$. With $p(x) = 2/x$, the integrating factor is

$$\int \frac{2}{x}\mathrm{d}x = 2\ln|x| = \ln x^2 \implies \mu(x) = x^2$$

Multiplying by the integrating factor yields

$$(x^2y)' = -\sin x \implies x^2y = \cos x + C \implies y(x) = \frac{\cos x}{x^2} + \frac{C}{x^2}$$

Applying the initial condition $y(\pi/2) = 0$ gives

$$y(\pi/2) = 0 = 0 + C/(\pi^2/4) \implies C = 0$$

so the solution is $y(x) = \frac{\cos x}{x^2}$.

ii. We need to determine when $y(x) = \frac{\cos x}{x^2} = -\frac{1}{\pi^2}$. By inspection, this occurs when $x = \pi$. ■