ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your instructor's name, (3) your lecture section number and (4) a grading table for ***six*** problems. Text books, class notes, cell phones and calculators are NOT permitted. A one page (letter sized **two sided**) crib sheet is allowed.

Problem 1 (40 points). Calculate the solutions to the following initial value problems.

(a) $y' = y^2$, $y(0) = 1$. At what time does the solution become infinite?

(b) $y' = -\frac{2}{t}y - 2$, $y(1) = 1$.

(c) $y' = \frac{1}{t}y + y^3$, $y(1) = 1$. Hint: Let $u = y^{-2}$, then find and solve ODE satisfied by u.

Solution:

(a) Solve by Separation of Variables:

$$y' = y^2 \;\Rightarrow\; \int y^{-2}dy = \int dt \;\Rightarrow\; -y^{-1} = t + C$$

$y(0) = 1$ implies $C = -1$ such that

$$y^{-1} = 1 - t \;\Rightarrow\; y(t) = \frac{1}{1-t}.$$

Solution becomes infinite at $t = 1$.

(b) By **Integrating Factors** $p(t) = 2/t$

$$\mu = e^{\int p(t)dt} = e^{\int \frac{2}{t}dt} = e^{2\ln t} = t^2.$$

Multiplication by I.F.

$$t^2\left(y' + \frac{2}{t}y\right] = -2t^2 \;\Rightarrow\; \left(t^2 y\right]' = -2t^2 \;\Rightarrow\; t^2 y(t) = -\frac{2}{3}t^3 + C \;\Rightarrow\; y(t) = -\frac{2}{3}t + \frac{C}{t^2}$$

Initial value $y(1) = 1$ implies $C = 5/3$, such that

$$y(t) = -\frac{2}{3}t + \frac{5}{3}\frac{1}{t^2}.$$

By two-stage **Euler-Lagrange** (i.e. Variation of Parameters).
(i) Homogeneous solution:

$$y' = -\frac{2}{t}y \;\Rightarrow\; \int \frac{1}{y}dy = -\int \frac{2}{t}dt \;\Rightarrow\; \ln|y| = -2\ln|t| + C \;\Rightarrow\; y_h(t) = \frac{C}{t^2}$$

(ii) Variation of Parameters:

$$y_p(t) = \frac{v(t)}{t^2} \;\Rightarrow\; \frac{1}{t^2}\frac{dv}{dt} = -2 \;\Rightarrow\; \frac{dv}{dt} = -2t^2 \;\Rightarrow\; v(t) = -\frac{2}{3}t^3 \;\Rightarrow\; y_p(t) = -\frac{2}{3}t$$

Again, initial value $y(1) = 1$ implies $C = 5/3$, such that

$$y(t) = -\frac{2}{3}t + \frac{5}{3}\frac{1}{t^2}.$$

(c) Let $u = y^{-2}$.The initial value $y(1) = 1$ converts to $u(1) = 1$. Then

$$\frac{du}{dt} = \frac{dy^{-2}}{dt} = -\frac{2}{y^3}\frac{dy}{dt} = -\frac{2}{y^3}\left(\frac{1}{t}y + y^3\right] = -\frac{2}{t}\frac{1}{y^2} - 2 = -\frac{2}{t}u - 2$$

Thus the IVP satisfied by u is

$$\frac{du}{dt} = -\frac{2}{t}u - 2, \quad u(1) = 1.$$

This is the same ODE as in part (b), so we find $u(t) = -\frac{2}{3}t + \frac{5}{3}\frac{1}{t^2}$. Since $u = y^{-2}$ we then find

$$y(t) = u^{-1/2} = \left[-\frac{2}{3}t + \frac{5}{3}\frac{1}{t^2}\right]^{-1/2}$$

Problem 2 (40 points) A tank initially contains 100 gallons of water in which 300 grams of an impurity are dissolved. Water containing the same impurity at a concentration of 2 grams per gallon enters the tank at a rate of 2 gallons per minute. Simultaneously, the well-mixed solution in the tank is pumped out at a rate of 4 gallons per minute. Let $A(t)$ denote the amount of the impurity in the tank at time t.

(a) What is the equation for the volume, $V(t)$, of fluid in the tank. At what time, t, does the tank become empty?
(b) Write an initial value problem that describes the amount of the impurity in the tank, $A(t)$.
(c) Solve the initial value problem from part (b).
(d) What is the amount of the impurity in the tank at $t = 25$ minutes.

Solution:

(a) The volume of solution in the tank as a function of time is

$$V(t) = 100 + (2 - 4)t = 100 - 2t$$

. The tank becomes empty when $t = 50$ minutes, so we are interested in values of t satisfying $0 \le t \le 50$.

(b) We let $A(t)$ be the amount of dissolved impurity at time t. Then

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \text{Rate in} - \text{Rate out} = \left(2\frac{\mathrm{g}}{\mathrm{gal}}\right)\left(2\frac{\mathrm{gal}}{\mathrm{min}}\right) - \left(\frac{A}{100-2t}\frac{\mathrm{g}}{\mathrm{gal}}\right)\left(4\frac{\mathrm{gal}}{\mathrm{min}}\right)$$

leading to the initial value problem (with first order linear differential equation)

$$\frac{\mathrm{d}A}{\mathrm{d}t} + \frac{4A}{100-2t} = 4 \qquad A(0) = 300$$

(c) We will solve with integrating factors. The integrating factor for this is obtained as

$$\int \frac{4}{100-2t}\,\mathrm{d}t = -2\ln|100-2t| \implies \mu(t) = (100-2t)^{-2}$$

yielding

$$\int \left[(100-2t)^{-2} A\right]' \mathrm{d}t = (100-2t)^{-2} A = \int 4(100-2t)^{-2}\,\mathrm{d}t = 2(100-2t)^{-1} + C$$

$$A(t) = 2(100-2t) + C(100-2t)^2$$

Applying the initial condition gives

$$A(0) = 200 + C(100^2) = 300 \implies C = \frac{1}{100}$$

and finally

$$\begin{aligned} A(t) &= 2(100-2t) + \frac{1}{100}(100-2t)^2 \\ &= (100-2t)\left[2 + \frac{1}{100}(100-2t)\right] \\ &= (100-2t)\left(3 - \frac{1}{50}t\right) \\ &= 300 - 8t + \frac{1}{25}t^2, \quad 0 \le t \le 50 \end{aligned}$$

(d) The amount of impurity in the water at $t = 25$ is given by

$$\begin{aligned} A(25) &= 2(100 - 2(25)) + \frac{1}{100}(100 - 2(25))^2 \\ &= 2(50) + \frac{1}{100}(50)^2 \\ &= 100 + 25 \\ &= 125 \text{ grams} \end{aligned}$$

Problem 3 (45 points) The following are unrelated.

(a) Calculate the inverse to $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$.

(b) In the following problems, $\mathbb{W}$ is a subset of a vector space $\mathbb{V}$. Determine whether $\mathbb{W}$ is a vector subspace of $\mathbb{V}$. If $\mathbb{W}$ is a vector subspace, prove it. Otherwise provide an explanation or counterexample to show why $\mathbb{W}$ does not form a vector subspace of $\mathbb{V}$. No credit will be given for responses without justification.

(i) $\mathbb{V} = \mathbb{M}_{33}$ (set of all 3×3 matrices). $\mathbb{W}$: All 3×3 matrices with $\text{Tr}(A) = 0$. (Recall $\text{Tr}(A)$ is the sum of the diagonal elements of a matrix A).

(ii) $\mathbb{V} = \mathbb{M}_{22}$ (set of all 2×2 matrices). $\mathbb{W}$: all 2×2 matrices with zero determinant.

(iii) $\mathbb{V} = \mathbb{P}_2$ (set of all polynomials of degree ≤ 2). $\mathbb{W}$: all quadratic polynomials, $p(x)$, with $x = 2$ as a root.

(c) (i) Give the definition for a set of functions $\{f_1, f_2, \ldots, f_n\}$ to be linearly independent.

(ii) Determine what the Wronskian tells about the linear dependence (or independence) of the functions $\{1, \cos^2 t, \sin^2 t\}$.

Solution:

(a) To find the inverse, we augment $\mathbf{A}$ with the identity matrix and then carry out row operations until we obtain the identity matrix in the left half, for example as follows

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3^* = R_3 + R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2^* = R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2/3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \\ 0 & 1 & 2 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \\ 0 & 1 & 0 & 1/3 & -2/3 & 1/3 \end{array}\right]$$

$$\xrightarrow{R_1^* = R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2/3 & 2/3 & -1/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \\ 0 & 1 & 0 & 1/3 & -2/3 & 1/3 \end{array}\right]$$

$$\xrightarrow{R_1^* = R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 1/3 & -2/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \\ 0 & 1 & 0 & 1/3 & -2/3 & 1/3 \end{array}\right]$$

$$\text{swap } R_2 \leftrightarrow R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 1/3 & -2/3 \\ 0 & 1 & 0 & 1/3 & -2/3 & 1/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right]$$

Therefore

$$\boxed{\mathbf{A}^{-1} = \begin{bmatrix} 1/3 & 1/3 & -2/3 \\ 1/3 & -2/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

(b) Vector spaces

(i) Consider general 3×3 matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

with zero trace, so $a_{11} + a_{22} + a_{33} = b_{11} + b_{22} + b_{33} =$. To verify that $\mathbb{W}$ is a subspace, then we verify the following

Closure under scalar multiplication: $\text{Tr}(cA) = c(a_{11} + a_{22} + a_{33}) = 0$

Closure under addition:

$$\begin{aligned} Tr(A+B) &= a_{11} + b_{11} + a_{22} + b_{22} + a_{33} + b_{33} \\ &= (a_{11} + a_{22} + a_{33}) + (b_{11} + b_{22} + b_{33}) \\ &= 0 + 0. \end{aligned}$$

Since the subset is closed under scalar multiplication and addition, then $\mathbb{W}$ is a vector subspace.

(ii) $\mathbb{W}$ is not a vector subspace. Consider

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

so M_1, $M_2 \in \mathbb{V}$. However, their sum

$$M_1 + M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

has determinant 1 so $\mathbb{W}$ is not closed under addition and hence not a vector

(iii) A quadratic polynomial with root $x = 2$ can be factored into the form

$$p(x) = k(x-2)(x-x_0),$$

where k and x_0 are arbitrary real constants. Now we need to verify closure under scalar multiplication and addition. Let $p_1(x) = k_1(x-2)(x-x_1)$ and $p_2(x) = k_2(x-2)(x-x_2)$.

Closure under scalar multiplication: if we multiply $p_1(x)$ by any constant c, we have

$$cp_1(x) = ck(x-2)(x-x1) \in \mathbb{W},$$

so the set is closed under scalar multiplication.

Closure under addition:

$$\begin{aligned} p_1(x) + p_2(x) &= k_1(x-2)(x-x_1) + k_2(x-2)(x-x_2) \\ &= (x-2)\left(k_1(x-x_1) + k_2(x-x_2)\right) \in \mathbb{W}, \end{aligned}$$

so the set is closed under addition.

Therefore $\mathbb{W}$ is a vector subspace of $\mathbb{V}$.

(c) (i) The functions are linearly independent if and only if the relation

$$c_1 f_1(t) + c_2 f_2(t) + \ldots + c_n f_n(t) = 0$$

implies that all the constants are zero.

(ii) The Wronskian is

$$\begin{aligned} W &= \begin{vmatrix} 1 & \sin^2 t & \cos^2 t \\ 0 & 2\sin t\cos t & -2\sin t\cos t \\ 0 & 2\left(\cos^2 t - \sin^2 t\right) & -2\left(\cos^2 t - \sin^2 t\right) \end{vmatrix} \\ &= 1 \begin{vmatrix} 2\sin t\cos t & -2\sin t\cos t \\ 2\left(\cos^2 t - \sin^2 t\right) & -2\left(\cos^2 t - \sin^2 t\right) \end{vmatrix} \\ &= 1\left[(2\sin t\cos t)(-2\left(\cos^2 t - \sin^2 t\right)) - (-2\sin t\cos t)\left(2\left(\cos^2 t - \sin^2 t\right)\right)\right] \\ &= 4\left[-\sin t\cos t(\cos^2 t - \sin^2 t) + \sin t\cos t(\cos^2 t - \sin^2 t)\right] \\ &= 0 \end{aligned}$$

Since the Wronskian is 0, then the test is inconclusive.

Problem 4 (40 points) Solve the initial value problem

$$y'' - 2y' + y = te^t$$
$$y(0) = 0 \quad y'(0) = 1,$$

using the **variation of parameters method**. At most half credit will be awarded if an alternative method is used.

Solution: We first find the homogeneous solution.

$$y'' - 2y' + y = 0,$$

which has the characteristic polynomial

$$r^2 - 2r + 1$$

upon seeking a solution e^{rt}. The roots of the characteristic polynomial are

$$r = 1 \quad \text{with multiplicity 2.}$$

Therefore the homogeneous solution is

$$y(t) = c_1 e^t + c_2 t e^t.$$

The two linearly independent solutions are

$$\begin{aligned} y_1 &= e^t, \\ y_2 &= te^t. \end{aligned}$$

The Wronskian of the linearly idependent solutions is

$$\begin{aligned} W[y_1, y_2] &= \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} \\ &= e^t\left(e^t + te^t\right) - te^{2t} \\ &= e^{2t} \end{aligned}$$

We seek the particular solution to be

$$y(t) = v_1(t)y_1 + v_2(t)y_2, \tag{1}$$

$$\begin{aligned} v_1' &= -\frac{y_2 f}{W[y_1, y_2]}, \\ v_2' &= \frac{y_1 f}{W[y_1, y_2]}, \end{aligned}$$

where $f = te^t$. Integrating, we find

$$\begin{aligned} v_1 &= -\int \frac{(te^t)(te^t)}{e^{2t}} dt \\ &= -\int t^2 dt \\ &= -\frac{1}{3}t^3 \end{aligned}$$

$$\begin{aligned} v_2 &= \int \frac{e^t(te^t)}{e^{2t}} dt \\ &= \int t dt \\ &= \frac{1}{2}t^2 \end{aligned}$$

Together, these give the particular solution

$$\begin{aligned} y_{\text{p}} &= -\frac{1}{3}t^3(e^t) + \frac{1}{2}t^2(te^t) \\ &= \frac{1}{6}t^3 e^t \end{aligned}$$

The general solution is then given by

$$y(t) = c_1 e^t + c_2 t e^t + \frac{1}{6} t^3 e^t$$

Applying the initial conditions, we have

$$y(0) = c_1 = 0$$
$$y'(0) = c_2 = 1,$$

therefore, the solution to the IVP is

$$\boxed{y(t) = te^t + \frac{1}{6} t^3 e^t}$$

Problem 5 (40 points) In this problem we will solve the initial value problem

$$y' - y = f(t), \qquad y(0) = -1.$$

Use **Laplace transforms** to solve the initial value problem with the following forcing functions:

(a) $f(t) = 5\cos(2t)$
(b) $f(t) = e^{-t}\ \text{step}(t-2)$

Solution:

(a) Taking the Laplace transform of all terms in the equation we have

$$\mathcal{L}\{y'\} = sY(s) + 1$$
$$\mathcal{L}\{y\} = Y(s)$$
$$\mathcal{L}\{5\cos 2t\} = \frac{5s}{s^2+4}$$

So the equation for $Y(s)$ is

$$sY(s) + 1 - Y(s) = \frac{5s}{s^2+4}$$
$$\implies Y(s) = \frac{5s}{(s-1)(s^2+4)} - \frac{1}{s-1}.$$

Performing a partial fraction decomposition on the first term, we find

$$Y(s) = \frac{-s+4}{s^2+4} + \frac{1}{s-1} - \frac{1}{s-1}$$

Canceling out the last two terms and rewriting we find

$$Y(s) = -\frac{s}{s^2+4} + \frac{4}{s^2+4}$$
$$= -\mathcal{L}\{\cos(2t)\} + 2\mathcal{L}\{\sin(2t)\}.$$

Therefore the solution to the IVP is

$$\boxed{y(t) = -\cos(2t) + 2\sin(2t)}$$

(b) Again we begin by taking the Laplace transform of both sides of the equation. The relevant transforms are

$$\mathcal{L}\{y'\} = sY(s) + 1$$
$$\mathcal{L}\{y\} = Y(s)$$
$$\mathcal{L}\left\{e^{-t}\text{step}(t-2)\right\} = e^{-2s}\mathcal{L}\left\{e^{-(t+2)}\right\} = e^{-2-2s}\frac{1}{s+1}$$

Upon taking a Laplace transform of the equation, we have

$$sY(s) + 1 - Y(s) = e^{-2-2s}\frac{1}{s+1}$$
$$\implies Y(s) = e^{-2-2s}\frac{1}{(s+1)(s-1)} - \frac{1}{s-1}.$$

Via a partial fraction decomposition, we have

$$Y(s) = e^{-2}\left(\frac{e^{-2s}}{2(s-1)} - \frac{e^{-2s}}{2(s+1)}\right) - \frac{1}{s-1}$$

Taking the inverse Laplace transform (using the tables and applying the shifting theorem) we have

$$\boxed{y(t) = e^{-2}\left(\frac{1}{2}e^{t-2} - \frac{1}{2}e^{-t+2}\right)\text{step}(t-2) - e^t}$$

Problem 6 (45 points)

(a) Consider the following system of coupled linear ODE's

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad \mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

(i) Find the eigenvalues of the matrix $\mathbf{A}$.
(ii) Find the eigenvectors of $\mathbf{A}$.
(iii) Show that all the eigenvectors computed in (ii) are linearly independent.
(iv) Find the general solution using the three linearly independent solutions associated with your computed eigenvalues and eigenvectors.
(v) What is the long time behavior of the general solution as $t \to \infty$.

(b) Now consider the linear system of ODE

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad \mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 1 & 1 & -2 \end{bmatrix}$$

which has the solution

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + c_3\mathbf{x}_3(t),$$

where the first two linearly independent solutions are

$$\mathbf{x_1}(t) = e^{-t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x_2}(t) = e^{-2t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

find the third linearly independent solution by solving a generalized eigenvalue problem with $\lambda = -2$.

Solution:

(a) (i) Matrix is lower triangular, therefore eigenvalues are $\lambda_1 = -1$, $\lambda_2 = \lambda_3 = -2$.

(ii) Solve $(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{v_i} = \mathbf{0}$ where $\mathbf{v_i} = (\alpha, \beta, \gamma)^T$ such that

$$\left[\begin{array}{ccc|c} -1-\lambda_i & 0 & 0 & 0 \\ 0 & -2-\lambda_i & 0 & 0 \\ 1 & 0 & -2-\lambda_i & 0 \end{array}\right].$$

Solution

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}_{\lambda_1=-1}, \quad \mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{\lambda_2=-2}, \quad \mathbf{v_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{\lambda_3=-2}.$$

Computation of eigenvectors

$$\lambda_1 = -1: \quad \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Free variable $\gamma = s$. Replace redundant row with this equation.

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & s \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s \end{array}\right] \Rightarrow \quad \mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$$\lambda_1 = -2: \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Free variables $\beta = r, \gamma = s$.Replace redundant row with these equations.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & 1 & s \end{array}\right] \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 0 \\ r \\ s \end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(iii) We determine linear independence by building a matrix whose columns are the eigenvectors computed in the previous part.

$$\det(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

(iv) General solution

$$\mathbf{x_g}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t) + c_3\mathbf{x_3}(t).$$

$$\mathbf{x_1}(t) = e^{-t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x_2}(t) = e^{-2t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x_3}(t) = e^{-2t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(v) As $t \to \infty$, all nontrivial solutions tend to the origin.

(b) The third solution is found by solving the generalized eigenvector problem

$$(\mathbf{A} + 2I)\mathbf{u} = \mathbf{v},$$

where $\mathbf{v}$ is the eigenvector that corresponds to the solution $\mathbf{x}_2$. The linear system for $\mathbf{u}$ is then

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & s \end{array}\right]$$

This then gives

$$\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

where s is a constant that can be taken to be $s = 0$. The third solution is then given by

$$\mathbf{x}_3(t) = e^{-2t}\left(t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

Table of Laplace transforms

$f(t)$	$F(s)$	s domain	$f(t)$	$F(s)$	s domain
1	$\frac{1}{s}$	$s > 0$	t^n	$\frac{n!}{s^{n+1}}$	$s > 0$, n a positive integer
e^{at}	$\frac{1}{s-a}$	$s > a$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$, n a positive integer
$\sin(bt)$	$\frac{b}{s^2+b^2}$	$s > 0$	$\cos(bt)$	$\frac{s}{s^2+b^2}$	$s > 0$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a$	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$\delta(t-c)$	e^{-cs}	$c \geq 0$, $s > 0$	$\text{step}(t-c)$	$\frac{e^{-sc}}{s}$	$c \geq 0$, $s > 0$
$f'(t)$	$sF(s) - f(0)$	depends on $f(t)$	$f(t-c)\text{step}(t-c)$	$e^{-cs}F(s)$	$c \geq 0$, $s > 0$

n^{th} order derivative: $\mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$