
On the front of your Bluebook write: (1) your name, (2) your instructor's name, (3) your lecture section number and (4) a grading table for ***six*** problems. Text books, class notes, cell phones and calculators are NOT permitted. A one page (letter sized **two sided**) crib sheet is allowed.

Problem 1 (40 points). Calculate the solutions to the following initial value problems.

(a) $y' = y^2$, $y(0) = 1$. At what time does the solution become infinite?

(b) $y' = -\dfrac{2}{t}y - 2$, $y(1) = 1$.

(c) $y' = \dfrac{1}{t}y + y^3$, $y(1) = 1$. Hint: Let $u = y^{-2}$, then find and solve ODE satisfied by u.

Problem 2 (40 points) A tank initially contains 100 gallons of water in which 300 grams of an impurity are dissolved. Water containing the same impurity at a concentration of 2 grams per gallon enters the tank at a rate of 2 gallons per minute. Simultaneously, the well-mixed solution in the tank is pumped out at a rate of 4 gallons per minute. Let $A(t)$ denote the amount of the impurity in the tank at time t.

(a) What is the equation for the volume, $V(t)$, of fluid in the tank. At what time, t, does the tank become empty?
(b) Write an initial value problem that describes the amount of the impurity in the tank, $A(t)$.
(c) Solve the initial value problem from part (b).
(d) What is the amount of the impurity in the tank at $t = 25$ minutes.

Problem 3 (45 points) The following are unrelated.

(a) Calculate the inverse to $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$.

(b) In the following problems, $\mathbb{W}$ is a subset of a vector space $\mathbb{V}$. Determine whether $\mathbb{W}$ is a vector subspace of $\mathbb{V}$. If $\mathbb{W}$ is a vector subspace, prove it. Otherwise provide an explanation or counterexample to show why $\mathbb{W}$ does not form a vector subspace of $\mathbb{V}$. No credit will be given for responses without justification.
  (i) $\mathbb{V} = \mathbb{M}_{33}$ (set of all 3×3 matrices). $\mathbb{W}$: All 3×3 matrices with $\mathrm{Tr}(A) = 0$. (Recall $\mathrm{Tr}(A)$ is the sum of the diagonal elements of a matrix A).
  (ii) $\mathbb{V} = \mathbb{M}_{22}$ (set of all 2×2 matrices). $\mathbb{W}$: all 2×2 matrices with zero determinant.
  (iii) $\mathbb{V} = \mathbb{P}_2$ (set of all polynomials of degree ≤ 2). $\mathbb{W}$: all quadratic polynomials, $p(x)$, with $x = 2$ as a root.

(c) (i) Give the definition for a set of functions $\{f_1, f_2, \ldots, f_n\}$ to be linearly independent.
  (ii) Determine what the Wronskian tells about the linear dependence (or independence) of the functions $\{1, \cos^2 t, \sin^2 t\}$.

Problem 4 (40 points) Solve the initial value problem

$$y'' - 2y' + y = te^t$$
$$y(0) = 0 \quad y'(0) = 1,$$

using the <u>**variation of parameters method**</u>. At most half credit will be awarded if an alternative method is used.

Problem 5 (40 points) In this problem we will solve the initial value problem

$$y' - y = f(t), \qquad y(0) = -1.$$

Use <u>**Laplace transforms**</u> to solve the initial value problem with the following forcing functions:

(a) $f(t) = 5\cos(2t)$
(b) $f(t) = e^{-t}\ \mathrm{step}(t - 2)$

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Problem 6 (45 points)

(a) Consider the following system of coupled linear ODE's

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad \mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

(i) Find the eigenvalues of the matrix $\mathbf{A}$.
(ii) Find the eigenvectors of $\mathbf{A}$.
(iii) Show that all the eigenvectors computed in (ii) are linearly independent.
(iv) Find the general solution using the three linearly independent solutions associated with your computed eigenvalues and eigenvectors.
(v) What is the long time behavior of the general solution as $t \to \infty$.

(b) Now consider the linear system of ODE

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad \mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 1 & 1 & -2 \end{bmatrix}$$

which has the solution

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + c_3\mathbf{x}_3(t),$$

where the first two linearly independent solutions are

$$\mathbf{x_1}(t) = e^{-t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x_2}(t) = e^{-2t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

find the third linearly independent solution by solving a generalized eigenvalue problem with $\lambda = -2$.

Table of Laplace transforms

$f(t)$	$F(s)$	s domain	$f(t)$	$F(s)$	s domain
1	$\frac{1}{s}$	$s > 0$	t^n	$\frac{n!}{s^{n+1}}$	$s > 0$, n a positive integer
e^{at}	$\frac{1}{s-a}$	$s > a$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$, n a positive integer
$\sin(bt)$	$\frac{b}{s^2+b^2}$	$s > 0$	$\cos(bt)$	$\frac{s}{s^2+b^2}$	$s > 0$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a$	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$\delta(t-c)$	e^{-cs}	$c \geq 0,\ s > 0$	$\text{step}(t-c)$	$\frac{e^{-sc}}{s}$	$c \geq 0,\ s > 0$
$f'(t)$	$sF(s) - f(0)$	depends on $f(t)$	$f(t-c)\text{step}(t-c)$	$e^{-cs}F(s)$	$c \geq 0,\ s > 0$

n^{th} order derivative: $\mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$