APPM 2360: Midterm exam 3

November 20, 2019

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your instructor's name, (3) your section (lecture) number and (4) a grading table. Text books, class notes, cell phones and calculators are NOT permitted. A one page (letter sized **one sided**) crib sheet is allowed.

Problem 1: (30 points) Consider the ordinary differential equation

$$y''' - y' = 0$$

(a) Determine the general solution for the differential equation.
(b) Verify that the solutions found in part (a) are linearly independent.

Solution:

(a) We seek solutions of the form $y = e^{rt}$ and insert the guess into the differential equation and find the characteristic polynomial

$$r^3 - r = 0 = r(r^2 - 1)$$

which has the characteristic roots $r = 0$, $r = 1$ and $r = -1$. The general solution then takes the form

$$y(t) = c_1 + c_2 e^{-t} + c_3 e^t$$

(b) We verify linear independence by computing the Wronskian of the solutions $\{1, e^t, e^{-t}\}$

$$\begin{aligned} W &= \begin{vmatrix} 1 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{vmatrix} \\ &= 1 \begin{vmatrix} e^t & -e^{-t} \\ e^t & e^{-t} \end{vmatrix} \\ &= 1\left[e^t(e^{-t}) - e^t(-e^{-t})\right] = 2. \end{aligned}$$

Since the Wronskain is nonzero, the set of solutions from part (a) is linearly independent.

Problem 2: (30 points) The following problems are unrelated.

(a) You received a toy spring with a 0.25 kg mass attached to it as a recent birthday present. Recognizing the toy as a harmonic oscillator and being mathematically inclined you, of course, have made some measurements:

- 8 newtons are required to stretch the spring 2 m;
- when moving at 6 m/sec, the mass experiences a damping force of 12 newtons.

  (i) One day, while playing with the toy, you pull the mass 1/2 m to the right of its equilibrium position and then push it to the left at 3 m/sec. Write, but **do not** solve, the initial value problem governing the motion of the oscillator if there are no external forces acting on it.
  (ii) Without finding $x(t)$, but providing mathematical justification, determine whether the oscillator is underdamped, overdamped or critically damped.

(b) Consider the harmonic oscillator described by the initial value problem

$$\ddot{x} + 2\dot{x} + 10x = 20, \quad x(0) = 2, \quad \dot{x}(0) = 3$$

  (i) Find the transient and steady-state solutions.
  (ii) If the damping is removed, find the frequency, ω_f, of a driving force of the form $F_0 \cos \omega_f t$ that will put the system into pure resonance.

Solution:

(a) (i) 8 newtons required to stretch the spring 2 meters: $k = F/x = 8\text{ N}/2\text{ m} = 4\text{ N/m}$
12 newtons when moving 6 m/sec: $b = F/\dot{x} = (12\text{ N})/(6\text{ m/sec}) = 2\text{ N/m/sec}$.
The initial value problem then becomes

$$m\ddot{x} + b\dot{x} + kx = 0 \implies \tfrac{1}{4}\ddot{x} + 2\dot{x} + 4x = 0$$

$$\text{with initial conditions} \quad x(0) = \tfrac{1}{2} \quad \text{and} \quad \dot{x}(0) = -3$$

(ii) $b^2 - 4mk = 2^2 - 4\left(\frac{1}{4}\right)4 = 0 \implies$ critically damped

(b) (i) Using the characteristic equation gives

$$r^2 + 2r + 10 = 0 \implies r = \frac{-2 \pm \sqrt{4 - 4(1)(10)}}{2} = \frac{-2 \pm \sqrt{-36}}{2} = -1 \pm 3i$$

yielding the solution to the homogeneous equation as

$$x_h(t) = e^{-t}\left(c_1 \cos 3t + c_2 \sin 3t\right)$$

The particular solution can be obtained by inspection as $x_p(t) = 2$, so that the solution is

$$x(t) = x_h(t) + x_p(t) = e^{-t}\left(c_1 \cos 3t + c_2 \sin 3t\right) + 2$$

Now apply the initial conditions

$$x(0) = e^0\left(c_1 \cos 0 + c_2 \sin 0\right) + 2 = 2 \implies c_1 = 0$$

so that

$$x(t) = c_2 e^{-t} \sin 3t + 2$$

and

$$\dot{x}(t) = c_2\left(3e^{-t}\cos 3t - e^{-t}\sin 3t\right)$$

$$\implies \dot{x}(0) = c_2\left(3e^0 \cos 0 - e^0 \sin 0\right) = 3 \implies c_2 = 1$$

and finally $x(t) = e^{-t}\sin 3t + 2$. The transient solution is $e^{-t}\sin 3t$ and the steady-state solution is 2.

(ii)

$$\omega_f = \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{10}$$

Problem 3: (30 points)

(a) (16 points) Use the *Method of Undetermined Coefficients* to write down the general form of the particular solution, y_p, for the differential equations given below but **do not solve for the coefficients**:

(i) $y'' - 3y' - 4y = -8e^{2t}\cos(3t)$ (ii) $y'' - 3y' - 4y = t^2 e^{-3t}\sin(t)$

(iii) $y'' - 3y' = 7 + 4y$ (iv) $y'' - 3y' - 4y = e^{-t}$

(b) (14 points) Now use the *Method of Undetermined Coefficients* to find a particular solution of the equation,

$$y'' - 3y' - 4y = e^{-t}$$

Solution:

(a) (i) $y_p = Ae^{2t}\cos(3t) + Be^{2t}\sin(3t)$

(ii) $y_p = (At^2 + Bt + C)e^{-3t}\sin(t) + (Dt^2 + Et + F)e^{-3t}\cos(t)$

(iii) $y_p = A$

(iv) $y_p = Ate^{-t}$, since $y = e^{-t}$ is a solution of the corresponding homogeneous equation. (Note $y = e^{4t}$ is also a solution of the corresponding homogeneous equation).

(b) If $y_p = Ate^{-t}$, then $y_p' = Ae^{-t} - Ate^{-t}$, and $y_p'' = -2Ae^{-t} + Ate^{-t}$, now substituting y_p into the equation $y'' - 3y' - 4y = e^{-t}$ yields $A = -1/5$ and so $y_p = -\dfrac{1}{5}te^{-t}$.

Problem 4: (30 points) Find the general solution $y(t)$ of the equation (defined e.g. on the time interval $0.1 \le t \le 1$ where it is non-singular)

$$ty'' - 2y' + \frac{2y}{t} = \frac{t^2}{\cos^2 t}$$

using the *method of variation of parameters.* Follow the steps below:

(a) Find and solve the characteristic equation for the corresponding homogeneous ODE. (*Hint:* Look for $y_h(t) = t^r$, r constant, and find all such r.)

(b) Write out the general solution $y_h(t)$ of the homogeneous ODE and the form of a particular solution $y_p(t)$ of the original ODE according to the *method of variation of parameters.*

(c) Find the Wronskian of the homogeneous solutions and check if they form a basis for the solution space of the homogeneous equation.

(d) Find the derivatives of the unknown functions in the expression for $y_p(t)$.

(e) Integrate the derivatives and write out a particular solution $y_p(t)$. (Recall that $(\tan t)' = 1/\cos^2 t$)

(f) Write out the general solution $y(t)$.

Solution:

(a) Look for $y_h = t^r$ and get the characteristic equation $r(r-1) - 2r + 2 = 0$ or

$$r^2 - 3r + 2 = 0,$$

so the roots are $r = 1$ and $r = 2$.

(b) Thus,

$$y_h(t) = C_1 t + C_2 t^2,$$

where C_1 and C_2 are arbitrary constants. Then look for $y_p(t)$ in the form

$$y_p(t) = v_1(t)t + v_2(t)t^2.$$

(c) Let $y_1(t) = t$ and $y_2(t) = t^2$. Then their Wronskian is

$$W = W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = t^2.$$

It is not identically zero so $\{y_1, y_2\}$ is a basis of homogeneous solutions.

(d) The forcing term to apply the standard formulas is obtained by dividing by the coefficient t of y'', it is $f(t) = t/\cos^2 t$. For $v_1'(t)$ and $v_2'(t)$, we get

$$v_1'(t) = \frac{\begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}}{W} = -\frac{y_2 f}{W} = -\frac{t}{\cos^2 t},$$

$$v_2'(t) = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix}}{W} = \frac{y_1 f}{W} = \frac{1}{\cos^2 t}.$$

(e) Integrating we get

$$v_2(t) = \tan t$$

and, integrating by parts,

$$v_1(t) = -\int \frac{t}{\cos^2 t} dt = -t \tan t + \int \tan t \, dt = -t \tan t - \ln|\cos t|.$$

Thus,

$$y_p(t) = t(-t \tan t - \ln|\cos t|) + t^2 \tan t = -t \ln|\cos t|.$$

(f) Finally,

$$y(t) = y_h(t) + y_p(t) = C_1 t + C_2 t^2 - t \ln|\cos t|.$$

Problem 5: (30 points) By means of the Laplace transform, solve the following two ODEs:

(a)

$$y'' + y = 1, \qquad y(0) = y'(0) = 0.$$

(b)

$$ty' + y = t, \qquad y(0) = 0.$$

Short table of Laplace Transforms: $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt$

$$\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathcal{L}\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathcal{L}\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n} \qquad \mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \qquad \mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0) \qquad \mathcal{L}\{tf'(t)\} = -F(s) - s\frac{dF(s)}{ds}$$

In this table, a and b can be any real numbers, and $n = 0, 1, 2, 3, \ldots$

Solution:

(a) Taking the Laplace transform gives $\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{1\}$. Using the bottom middle and top left (case $a = n = 0$) table entries gives

$$s^2 Y(s) + Y(s) = \frac{1}{s}$$

and thus

$$Y(s) = \frac{1}{s(s^2+1)},$$

which can be split in partial fractions as

$$Y(s) = \frac{1}{s} - \frac{s}{s^2+1}.$$

The top left and top middle table entries give now

$$y(t) = 1 - \cos t.$$

(b) Taking the Laplace transform gives $\mathcal{L}\{ty'\} + \mathcal{L}\{y\} = \mathcal{L}\{t\}$, i.e. (using the bottom right and top left table entries)

$$-Y(s) - sY'(s) + Y(s) = \frac{1}{s^2},$$

from which follows $Y'(s) = -1/s^3$ and

$$Y(s) = \frac{1}{2s^2} + C,$$

where $C = 0$ (since we need $Y(s) \to 0$ as $s \to \infty$). From the top left table entry ($a = 0$, $n = 1$) it follows that

$$y(t) = \frac{t}{2}.$$