APPM 2360: Midterm exam 3
November 20, 2019

On the front of your Bluebook write: (1) your name, (2) your instructor's name, (3) your section (lecture) number and (4) a grading table. Text books, class notes, cell phones and calculators are NOT permitted. A one page (letter sized **one sided**) crib sheet is allowed.

Problem 1: (30 points) Consider the ordinary differential equation

$$y''' - y' = 0$$

(a) Determine the general solution for the differential equation.
(b) Verify that the solutions found in part (a) are linearly independent.

Problem 2: (30 points) The following problems are unrelated.

(a) You received a toy spring with a 0.25 kg mass attached to it as a recent birthday present. Recognizing the toy as a harmonic oscillator and being mathematically inclined you, of course, have made some measurements:
- 8 newtons are required to stretch the spring 2 m;
- when moving at 6 m/sec, the mass experiences a damping force of 12 newtons.

  (i) One day, while playing with the toy, you pull the mass 1/2 m to the right of its equilibrium position and then push it to the left at 3 m/sec. Write, but **do not** solve, the initial value problem governing the motion of the oscillator if there are no external forces acting on it.
  (ii) Without finding $x(t)$, but providing mathematical justification, determine whether the oscillator is underdamped, overdamped or critically damped.

(b) Consider the harmonic oscillator described by the initial value problem

$$\ddot{x} + 2\dot{x} + 10x = 20, \quad x(0) = 2, \quad \dot{x}(0) = 3$$

  (i) Find the transient and steady-state solutions.
  (ii) If the damping is removed, find the frequency, ω_f, of a driving force of the form $F_0 \cos \omega_f t$ that will put the system into pure resonance.

Problem 3: (30 points)

(a) (16 points) Use the *Method of Undetermined Coefficients* to write down the general form of the particular solution, y_p, for the differential equations given below but **do not solve for the coefficients**:

(i) $y'' - 3y' - 4y = -8e^{2t}\cos(3t)$ (ii) $y'' - 3y' - 4y = t^2 e^{-3t}\sin(t)$

(iii) $y'' - 3y' = 7 + 4y$ (iv) $y'' - 3y' - 4y = e^{-t}$

(b) (14 points) Now use the *Method of Undetermined Coefficients* to find a particular solution of the equation,

$$y'' - 3y' - 4y = e^{-t}$$

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Problem 4: (30 points) Find the general solution $y(t)$ of the equation (defined e.g. on the time interval $0.1 \leq t \leq 1$ where it is non-singular)

$$ty'' - 2y' + \frac{2y}{t} = \frac{t^2}{\cos^2 t}$$

using the *method of variation of parameters*. Follow the steps below:

(a) Find and solve the characteristic equation for the corresponding homogeneous ODE. (*Hint:* Look for $y_h(t) = t^r$, r constant, and find all such r.)

(b) Write out the general solution $y_h(t)$ of the homogeneous ODE and the form of a particular solution $y_p(t)$ of the original ODE according to the *method of variation of parameters*.

(c) Find the Wronskian of the homogeneous solutions and check if they form a basis for the solution space of the homogeneous equation.

(d) Find the derivatives of the unknown functions in the expression for $y_p(t)$.

(e) Integrate the derivatives and write out a particular solution $y_p(t)$. (Recall that $(\tan t)' = 1/\cos^2 t$)

(f) Write out the general solution $y(t)$.

Problem 5: (30 points) By means of the Laplace transform, solve the following two ODEs:

(a)

$$y'' + y = 1, \qquad y(0) = y'(0) = 0.$$

(b)

$$ty' + y = t, \qquad y(0) = 0.$$

Short table of Laplace Transforms: $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt$

$$\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathcal{L}\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathcal{L}\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n} \qquad \mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \qquad \mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0) \qquad \mathcal{L}\{tf'(t)\} = -F(s) - s\frac{dF(s)}{ds}$$

In this table, a and b can be any real numbers, and $n = 0, 1, 2, 3, \dots$