PROBLEM 1 (30 points): The following problems are not related.

(a) For the following matrices, are the products $\mathbf{CD}$ and/or $\mathbf{DC}$ defined? If so, compute them. If not, explain why not.

$$\mathbf{C} = \begin{bmatrix} 2 & -1 \\ 6 & 5 \\ -1 & 0 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{bmatrix}$$

(b) Find all the 2×2 matrices that commute with the given matrix $\begin{bmatrix} 2 & 0 \\ k & -2 \end{bmatrix}$, where $k \neq 0$ is real. (Matrices $\mathbf{A}$ and $\mathbf{B}$ commute if $\mathbf{AB} = \mathbf{BA}$.)

(c) Let $\mathbb{W}$ be the subset of $\mathbb{M}_{22}$ with elements of the form $\vec{\mathbf{v}} = \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix}$ where k is a real number. Is $\mathbb{W}$ a subspace? A simple yes or no answer will receive no credit. Show your work.

SOLUTION:

(a) Both products $\mathbf{CD}$ and $\mathbf{DC}$ are defined since $\mathbf{C}$ is 3×2 and $\mathbf{D}$ is 2×3.

$$\mathbf{CD} = \begin{bmatrix} 0 & -4 & 5 \\ 16 & -12 & 23 \\ -1 & 2 & -3 \end{bmatrix}, \qquad \mathbf{DC} = \begin{bmatrix} -13 & -11 \\ 3 & -2 \end{bmatrix}$$

(b) Let the sought after matrix be

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 0 \\ k & -2 \end{bmatrix} = \begin{bmatrix} 2a + kb & -2b \\ 2c + kd & -2d \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & 0 \\ k & -2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ ka - 2c & kb - 2d \end{bmatrix}.$$

Equating the right-hand sides of the products, we get $2a + kb = 2a$, $-2b = 2b$, $2c + kd = ka - 2c$ and $-2d = kb - 2d$. This implies $b = 0$ and the only other restriction is $k(a - d) = 4c$, *i.e.*, all matrices of the form

$$\begin{bmatrix} a & 0 \\ k(a-d)/4 & d \end{bmatrix},$$

where a and d are arbitrary, will commute with the given matrix.

(c) The zero vector $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is in the set $\mathbb{W}$. Now suppose $\vec{\mathbf{u}} = \begin{bmatrix} 0 & p \\ p & 0 \end{bmatrix}$ and $\vec{\mathbf{v}} = \begin{bmatrix} 0 & q \\ q & 0 \end{bmatrix}$ are in $\mathbb{W}$ and let a and b be real numbers. Then

$$a\vec{\mathbf{u}} + b\vec{\mathbf{v}} = a\begin{bmatrix} 0 & p \\ p & 0 \end{bmatrix} + b\begin{bmatrix} 0 & q \\ q & 0 \end{bmatrix} = \begin{bmatrix} 0 & ap \\ ap & 0 \end{bmatrix} + \begin{bmatrix} 0 & bq \\ bq & 0 \end{bmatrix} = \begin{bmatrix} 0 & ap + bq \\ ap + bq & 0 \end{bmatrix} \in \mathbb{W} \implies \mathbb{W} \text{ is a subspace.}$$

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PROBLEM 2: (30 points) These problems are not related.

(a) Let $\mathbf{C}$ and $\mathbf{D}$ be invertible matrices. Solve for $\vec{\mathbf{x}}$ if $\mathbf{C}(\mathbf{DC})^{-1}\vec{\mathbf{x}} = \vec{\mathbf{y}}$. Be sure to simplify your answer.

(b) For which value(s) of k is the matrix $\begin{bmatrix} 1 & 0 & k \\ 0 & k & 1 \\ k & 0 & 4 \end{bmatrix}$ invertible?

(c) Consider the matrix $\mathbf{B} = \begin{bmatrix} 2 & 4 & 5 \\ 1 & 2 & 3 \\ 3 & 5 & 6 \end{bmatrix}$.

(i) Use Gauss-Jordan Reduction to find $\mathbf{B}^{-1}$.

(ii) Use your answer to part (i) to find the solution(s) of the linear system

$$\begin{aligned} 2x_1 + 4x_2 + 5x_3 &= 3 \\ x_1 + 2x_2 + 3x_3 &= -1 \\ 3x_1 + 5x_2 + 6x_3 &= 2 \end{aligned}$$

(iii) Find the solution(s) to the associated homogeneous system in part (ii).

SOLUTION:

(a)

$$\vec{\mathbf{x}} = \left[\mathbf{C}(\mathbf{DC})^{-1}\right]^{-1}\vec{\mathbf{y}} = \left[(\mathbf{DC})^{-1}\right]^{-1}\mathbf{C}^{-1}\vec{\mathbf{y}} = \left(\mathbf{C}^{-1}\mathbf{D}^{-1}\right)^{-1}\mathbf{C}^{-1}\vec{\mathbf{y}}$$
$$= (\mathbf{D}^{-1})^{-1}(\mathbf{C}^{-1})^{-1}\mathbf{C}^{-1}\vec{\mathbf{y}} = \mathbf{D}(\mathbf{C}\mathbf{C}^{-1})\vec{\mathbf{y}} = \mathbf{D}\mathbf{I}\vec{\mathbf{y}} = \mathbf{D}\vec{\mathbf{y}}$$

(b) The matrix is invertible if its determinant is nonzero.

$$\begin{vmatrix} 1 & 0 & k \\ 0 & k & 1 \\ k & 0 & 4 \end{vmatrix} = 4k - k^3 = k(4-k^2) = k(k+2)(k-2)$$

This will be nonzero if $k \neq 0$ or $k \neq \pm 2$.

(c) Augment $\mathbf{B}$ with the 3×3 identity matrix and perform the appropriate elementary row operations

(i)

$$\left[\begin{array}{ccc|ccc} 2 & 4 & 5 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 4 & 5 & 1 & 0 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3^*=-3R_1+R_3]{R_2^*=-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 0 \\ 0 & -1 & -3 & 0 & -3 & 1 \end{array}\right]$$

$$\xrightarrow[R_3^*=-R_2]{R_2^*=-R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 3 & -1 \\ 0 & 0 & 1 & -1 & 2 & 0 \end{array}\right] \xrightarrow[R_2^*=-3R_3+R_2]{R_1^*=-3R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 3 & -5 & 0 \\ 0 & 1 & 0 & 3 & -3 & -1 \\ 0 & 0 & 1 & -1 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{R_1^*=-2R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 1 & 2 \\ 0 & 1 & 0 & 3 & -3 & -1 \\ 0 & 0 & 1 & -1 & 2 & 0 \end{array}\right] \Longrightarrow \mathbf{B}^{-1} = \begin{bmatrix} -3 & 1 & 2 \\ 3 & -3 & -1 \\ -1 & 2 & 0 \end{bmatrix}$$

(ii) The linear system can be written as a matrix equation as

$$\begin{bmatrix} 2 & 4 & 5 \\ 1 & 2 & 3 \\ 3 & 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \text{ or } \mathbf{B}\vec{\mathbf{x}} = \vec{\mathbf{b}} \Longrightarrow \vec{\mathbf{x}} = \mathbf{B}^{-1}\vec{\mathbf{b}} \Longrightarrow \vec{\mathbf{x}} = \begin{bmatrix} -3 & 1 & 2 \\ 3 & -3 & -1 \\ -1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ 10 \\ -5 \end{bmatrix}$$

(iii) In this case we have

$$\vec{\mathbf{x}} = \begin{bmatrix} -3 & 1 & 2 \\ 3 & -3 & -1 \\ -1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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PROBLEM 3: (30 points) Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$.

(a) Evaluate $|\mathbf{A}|$ by expanding and using the appropriate cofactor formula.

(b) Bring the matrix $\mathbf{A}$ to RREF (Reduced Row Echelon Form) and deduce from this process the value of $|\mathbf{A}|$.

(c) Consider the linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ where $\mathbf{A}$ is the same matrix as above, and $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\vec{\mathbf{b}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Use Cramer's Rule to determine the second component, x_2, in the solution vector $\vec{\mathbf{x}}$.

SOLUTION:

(a) Expand down the first column

$$|\mathbf{A}| = (1)(-1)^{1+1}\begin{vmatrix} -1 & 0 \\ 1 & 3 \end{vmatrix} + (2)(-1)^{2+1}\begin{vmatrix} -1 & 1 \\ 1 & 3 \end{vmatrix} + (0)(-1)^{3+1}\begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} = (1)(-3) - 2(-4) = 5$$

(b)

$$\overbrace{\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & 3 \end{bmatrix}}^{\mathbf{A}} \xrightarrow{R_2^*=-2R_1+R_2} \overbrace{\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & 3 \end{bmatrix}}^{\mathbf{B}} \xrightarrow{R_3^*=-R_2+R_3} \overbrace{\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{bmatrix}}^{\mathbf{C}} \xrightarrow{R_1^*=R_1+R_2} \overbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{bmatrix}}^{\mathbf{D}}$$

$$\xrightarrow{R_3^*=\frac{1}{5}R_3} \overbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}}^{\mathbf{E}} \xrightarrow[R_2^*=2R_3+R_2]{R_1^*=R_1+R_3} \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\mathbf{I}}$$

From these operations we have

$$|\mathbf{B}| = |\mathbf{A}|, \ \ |\mathbf{C}| = |\mathbf{B}|, \ \ |\mathbf{D}| = |\mathbf{C}|, \ \ |\mathbf{E}| = \tfrac{1}{5}|\mathbf{D}|, \ \ |\mathbf{I}| = |\mathbf{E}|$$

$$\implies |\mathbf{I}| = |\mathbf{E}| = \tfrac{1}{5}|\mathbf{D}| = \tfrac{1}{5}|\mathbf{C}| = \tfrac{1}{5}|\mathbf{B}| = \tfrac{1}{5}|\mathbf{A}| \implies |\mathbf{A}| = 5|\mathbf{I}| = 5(1) = 5$$

(c)

$$x_2 = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{vmatrix}}{|\mathbf{A}|} = \frac{(3)(-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}}{5} = -\frac{3}{5}$$

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PROBLEM 4: (30 points) Let $\mathbb{PE}_4$ define the vector space of all even symmetric polynomials of degree ≤ 4. All vector elements $\vec{\mathbf{v}}$ in $\mathbb{PE}_4$ therefore have the form

$$\vec{\mathbf{v}} = a_0 + a_2 t^2 + a_4 t^4.$$

As an alternative interpretation, note that every vector $\vec{\mathbf{v}} \in \mathbb{PE}_4$ can be thought of as a 3-vector $\vec{\mathbf{v}} = [a_0, a_2, a_4]^{\mathrm{T}}$.

(a) What is the dimension of $\mathbb{PE}_4$?

(b) Write the corresponding 3-vector forms for

$$\vec{\mathbf{v}}_1 = 1 + t^2 + t^4, \quad \vec{\mathbf{v}}_2 = -1 + t^4, \quad \vec{\mathbf{v}}_3 = -2 + t^2 + 4t^4$$

(c) Do the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ above form a basis for $\mathbb{PE}_4$? No credit will be given for a simple yes or no answer. Show your work.

(d) Given $\vec{\mathbf{v}} = t^4$, is $\vec{\mathbf{v}} \in \mathrm{Span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$? No credit will be given for a simple yes or no answer. Show your work.

SOLUTION:

(a) dim $\mathbb{PE}_4 = 3$

(b) $\vec{\mathbf{v}}_1 = [1, 1, 1]^{\mathrm{T}}, \quad \vec{\mathbf{v}}_2 = [-1, 0, 1]^{\mathrm{T}}, \quad \vec{\mathbf{v}}_3 = [-2, 1, 4]^{\mathrm{T}}.$

(c) No, the polynomials are linearly dependent:

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 2 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Not needed but

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & a_0 \\ 1 & 0 & 1 & a_2 \\ 1 & 1 & 4 & a_4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & a_0 \\ 0 & 1 & 3 & a_2 - a_0 \\ 0 & 2 & 6 & a_4 - a_0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & a_0 \\ 0 & 1 & 3 & a_2 - a_0 \\ 0 & 0 & 0 & a_4 - 2a_2 + a_0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & a_2 \\ 0 & 1 & 3 & a_2 - a_0 \\ 0 & 0 & 0 & a_4 - 2a_2 + a_0 \end{array}\right]$$

(d) No, $\vec{\mathbf{v}} = t^4 \notin \text{Span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 4 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 2 & 6 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

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PROBLEM 5: (30 points) The following problems are unrelated

(a) Compute the eigenvalues of the matrix $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$.

(b) Consider the 3×3 matrix $\mathbf{B} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 4 & 1 \\ 2 & -4 & 0 \end{bmatrix}$, with $\lambda = 2$ among its eigenvalues (you do not need to verify this). Compute the eigenvector(s) of $\mathbf{B}$ corresponding to $\lambda = 2$.

SOLUTION:

(a) We compute eigenvalues by computing values of λ such that

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

where $\mathbf{I}$ is the 2×2 identity matrix. Computing the characteristic polynomial, we have

$$P(\lambda) = \begin{vmatrix} 3-\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda) - 2 = \lambda^2 - 4\lambda + 1$$

The roots of $P(\lambda) = 0$ are given by

$$\lambda_\pm = \frac{4 \pm \sqrt{16 - 4(1)(1)}}{2} = 2 \pm \sqrt{3}$$

(b) To compute the eigenvectors corresponding to the eigenvalue $\lambda = 2$, we need find the solutions of

$$(\mathbf{B} - 2\mathbf{I})\vec{\mathbf{v}} = 0$$

We find $\vec{\mathbf{v}}$ by row reducing the augmented matrix

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & -4 & -2 & 0 \end{array}\right] \quad \begin{array}{c} \\ R_2^* = -1R_1 + R_2 \\ R_3^* = 2R_1 + R_3 \end{array} \left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1^* = -R_1 \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The components of the eigenvector $\vec{\mathbf{v}} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ obey the relationship $v_1 = 2v_2 + v_3$, where v_2 and v_3 are free variables. The solution space of the problem $(\mathbf{B} - 2\mathbf{I})\vec{\mathbf{v}} = 0$ is therefore given by

$$\vec{\mathbf{v}} = \begin{bmatrix} 2r + s \\ r \\ s \end{bmatrix} = \begin{bmatrix} 2r \\ r \\ 0 \end{bmatrix} + \begin{bmatrix} s \\ 0 \\ s \end{bmatrix} = r \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Thus, we have two linearly independent eigenvectors

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

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