ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your instructor’s name, (3) your lecture section number and (4) a grading table. Textbooks, class notes, cell phones and calculators are NOT permitted. A one page (letter sized, 1 side only) crib sheet is allowed. The exam has five (5) problems and is worth 150 points. Show all of your work and simplify and box your final answers.

PROBLEM 1 (30 points): The following problems are not related.

(a) For the following matrices, are the products $CD$ and/or $DC$ defined? If so, compute them. If not, explain why not.

$$C = \begin{bmatrix} 2 & -1 \\ 6 & 5 \\ -1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{bmatrix}$$

(b) Find all the $2 \times 2$ matrices that commute with the given matrix $\begin{bmatrix} 2 & 0 \\ k & -2 \end{bmatrix}$, where $k \neq 0$ is real. (Matrices $A$ and $B$ commute if $AB = BA$.)

(c) Let $\mathcal{W}$ be the subset of $M_{2,2}$ with elements of the form $\vec{v} = \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix}$ where $k$ is a real number. Is $\mathcal{W}$ a subspace? A simple yes or no answer will receive no credit. Show your work.

PROBLEM 2: (30 points) These problems are not related.

(a) Let $C$ and $D$ be invertible matrices. Solve for $\vec{x}$ if $C(DC)^{-1} \vec{x} = \vec{y}$. Be sure to simplify your answer.

(b) For which value(s) of $k$ is the matrix $\begin{bmatrix} 1 & 0 & k \\ 0 & k & 1 \\ k & 0 & 4 \end{bmatrix}$ invertible?

(c) Consider the matrix $B = \begin{bmatrix} 2 & 4 & 5 \\ 1 & 2 & 3 \\ 3 & 5 & 6 \end{bmatrix}$.

(i) Use Gauss-Jordan Reduction to find $B^{-1}$.

(ii) Use your answer to part (i) to find the solution(s) of the linear system

$$\begin{align*}
2x_1 + 4x_2 + 5x_3 &= 3 \\
x_1 + 2x_2 + 3x_3 &= -1 \\
3x_1 + 5x_2 + 6x_3 &= 2
\end{align*}$$

(iii) Find the solution(s) to the associated homogeneous system in part (ii).

PROBLEM 3: (30 points) Consider the matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$.

(a) Evaluate $|A|$ by expanding and using the appropriate cofactor formula.

(b) Bring the matrix $A$ to RREF (Reduced Row Echelon Form) and deduce from this process the value of $|A|$.

(c) Consider the linear system $A\vec{x} = \vec{b}$ where $A$ is the same matrix as above, and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Use Cramer’s Rule to determine the second component, $x_2$, in the solution vector $\vec{x}$.

CONTINUED ON BACK
**PROBLEM 4:** (30 points) Let $\mathbb{P}_4$ define the vector space of all even symmetric polynomials of degree $\leq 4$. All vector elements $\vec{v}$ in $\mathbb{P}_4$ therefore have the form

$$\vec{v} = a_0 + a_2 t^2 + a_4 t^4.$$ 

As an alternative interpretation, note that every vector $\vec{v} \in \mathbb{P}_4$ can be thought of as a 3-vector $\vec{v} = [a_0, a_2, a_4]^T$.

(a) What is the dimension of $\mathbb{P}_4$?

(b) Write the corresponding 3-vector forms for

$$\vec{v}_1 = 1 + t^2 + t^4, \quad \vec{v}_2 = -1 + t^4, \quad \vec{v}_3 = -2 + t^2 + 4t^4$$

(c) Do the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ above form a basis for $\mathbb{P}_4$? No credit will be given for a simple yes or no answer. Show your work.

(d) Given $\vec{v} = t^4$, is $\vec{v} \in \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$? No credit will be given for a simple yes or no answer. Show your work.

**PROBLEM 5:** (30 points) The following problems are unrelated

(a) Compute the eigenvalues of the matrix $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$.

(b) Consider the $3 \times 3$ matrix $B = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 4 & 1 \\ 2 & -4 & 0 \end{bmatrix}$, with $\lambda = 2$ among its eigenvalues (you do not need to verify this). Compute the eigenvector(s) of $B$ corresponding to $\lambda = 2$. 
