ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your instructor's name, (3) your lecture section number and (4) a grading table. Textbooks, class notes, cell phones and calculators are NOT permitted. A one page (letter sized, 1 side only) crib sheet is allowed. The exam has five (5) problems and is worth 150 points. Show all of your work and simplify and box your final answers.

PROBLEM 1 (30 points): The following problems are not related.

(a) For the following matrices, are the products $\mathbf{CD}$ and/or $\mathbf{DC}$ defined? If so, compute them. If not, explain why not.

$$\mathbf{C} = \begin{bmatrix} 2 & -1 \\ 6 & 5 \\ -1 & 0 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{bmatrix}$$

(b) Find all the 2×2 matrices that commute with the given matrix $\begin{bmatrix} 2 & 0 \\ k & -2 \end{bmatrix}$, where $k \neq 0$ is real. (Matrices $\mathbf{A}$ and $\mathbf{B}$ commute if $\mathbf{AB} = \mathbf{BA}$.)

(c) Let $\mathbb{W}$ be the subset of $\mathbb{M}_{22}$ with elements of the form $\vec{\mathbf{v}} = \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix}$ where k is a real number. Is $\mathbb{W}$ a subspace? A simple yes or no answer will receive no credit. Show your work.

PROBLEM 2: (30 points) These problems are not related.

(a) Let $\mathbf{C}$ and $\mathbf{D}$ be invertible matrices. Solve for $\vec{\mathbf{x}}$ if $\mathbf{C}(\mathbf{DC})^{-1}\vec{\mathbf{x}} = \vec{\mathbf{y}}$. Be sure to simplify your answer.

(b) For which value(s) of k is the matrix $\begin{bmatrix} 1 & 0 & k \\ 0 & k & 1 \\ k & 0 & 4 \end{bmatrix}$ invertible?

(c) Consider the matrix $\mathbf{B} = \begin{bmatrix} 2 & 4 & 5 \\ 1 & 2 & 3 \\ 3 & 5 & 6 \end{bmatrix}$.

(i) Use Gauss-Jordan Reduction to find $\mathbf{B}^{-1}$.

(ii) Use your answer to part (i) to find the solution(s) of the linear system

$$\begin{aligned} 2x_1 + 4x_2 + 5x_3 &= 3 \\ x_1 + 2x_2 + 3x_3 &= -1 \\ 3x_1 + 5x_2 + 6x_3 &= 2 \end{aligned}$$

(iii) Find the solution(s) to the associated homogeneous system in part (ii).

PROBLEM 3: (30 points) Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$.

(a) Evaluate $|\mathbf{A}|$ by expanding and using the appropriate cofactor formula.

(b) Bring the matrix $\mathbf{A}$ to RREF (Reduced Row Echelon Form) and deduce from this process the value of $|\mathbf{A}|$.

(c) Consider the linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ where $\mathbf{A}$ is the same matrix as above, and $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\vec{\mathbf{b}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Use Cramer's Rule to determine the second component, x_2, in the solution vector $\vec{\mathbf{x}}$.

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PROBLEM 4: (30 points) Let $\mathbb{PE}_4$ define the vector space of all even symmetric polynomials of degree ≤ 4. All vector elements $\vec{\mathbf{v}}$ in $\mathbb{PE}_4$ therefore have the form

$$\vec{\mathbf{v}} = a_0 + a_2 t^2 + a_4 t^4.$$

As an alternative interpretation, note that every vector $\vec{\mathbf{v}} \in \mathbb{PE}_4$ can be thought of as a 3-vector $\vec{\mathbf{v}} = [a_0, a_2, a_4]^{\mathrm{T}}$.

(a) What is the dimension of $\mathbb{PE}_4$?

(b) Write the corresponding 3-vector forms for

$$\vec{\mathbf{v}}_1 = 1 + t^2 + t^4, \quad \vec{\mathbf{v}}_2 = -1 + t^4, \quad \vec{\mathbf{v}}_3 = -2 + t^2 + 4t^4$$

(c) Do the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ above form a basis for $\mathbb{PE}_4$? No credit will be given for a simple yes or no answer. Show your work.

(d) Given $\vec{\mathbf{v}} = t^4$, is $\vec{\mathbf{v}} \in \mathrm{Span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$? No credit will be given for a simple yes or no answer. Show your work.

PROBLEM 5: (30 points) The following problems are unrelated

(a) Compute the eigenvalues of the matrix $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$.

(b) Consider the 3×3 matrix $\mathbf{B} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 4 & 1 \\ 2 & -4 & 0 \end{bmatrix}$, with $\lambda = 2$ among its eigenvalues (you do not need to verify this). Compute the eigenvector(s) of $\mathbf{B}$ corresponding to $\lambda = 2$.