# APPM 2360: Midterm Exam 1 September 25, 2019 5-6:30 pm

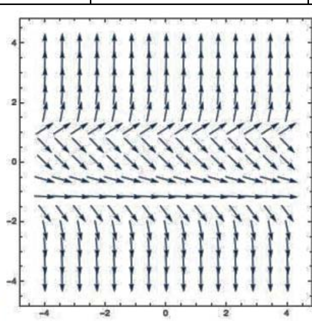
ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your instructor's name, (3) your recitation section number and (4) a grading table. Text books, class notes, cell phones and calculators are NOT permitted. A one page (letter sized, 1 side only) crib sheet is allowed. Each problem is worth 30 points. Box your answers.

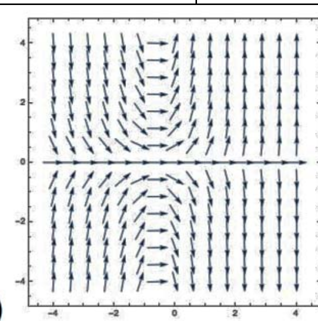
**Problem 1:**

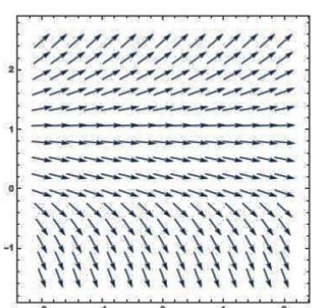
Instruction: The solution to this problem should be in a form of a (6 row)$\times$(4 column) table in your blue book (no need to copy the leading row and column with labels from the table below). In the top row of your table, you should give "(a),...,(d)" in the appropriate order, and all other table entries should be either "Y" for yes, "N" for no, or "-" for not applicable. Grading points will be awarded only according to the entries in this table.

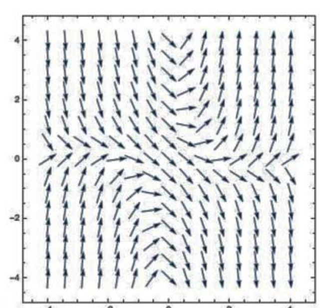
For each of the four ordinary differential equations specified in the top row, fill out your table copy as instructed above. The last row is about solutions satisfying initial condition $y(t_0) = y_0$. The region $R$ is $R = \{(t, y) \mid -4 < t < 4, -4 < y < 4\}$.

| | $y' = ty - \cos(t)$ | $y' = (y+1)^2(y-1)$ | $y' = y - \sqrt{\lvert y \rvert}$ | $y' = e^t y + ty$ |
|---|---|---|---|---|
| Match the direction fields from Fig. 1 | | | | |
| Is the equation linear? | | | | |
| If linear, is it homogeneous? | | | | |
| Is the equation separable? | | | | |
| If $y_1(t)$ and $y_2(t)$ solve the ODE, does $y_3(t) = y_1(t) + y_2(t)$ also solve the ODE? | | | | |
| Can you guarantee solutions are unique for all $(t_0, y_0)$ in $R$? | | | | |


(a)


(b)


(c)


(d)

**Solution:**

| | $y' = ty - \cos(t)$ | $y' = (y+1)^2(y-1)$ | $y' = y - \sqrt{\lvert y \rvert}$ | $y' = e^t y + ty$ |
|---|---|---|---|---|
| Match the direction fields from Fig. 1 | (d) | (a) | (c) | (b) |
| Is the equation linear? | Y | N | N | Y |
| If linear, is it homogeneous? | N | - | - | Y |
| Is the equation separable? | N | Y | Y | Y |
| If $y_1(t)$ and $y_2(t)$ solve the ODE, does $y_3(t) = y_1(t) + y_2(t)$ also solve the ODE? | N | N | N | Y |
| Can you guarantee solutions are unique for all $(t_0, y_0)$ in *R*? | Y | Y | N | Y |

**Problem 2:** A friend of yours is undergoing surgery and must be anesthetized. The anesthesiologist knows that your friend will be "under" when the concentration of sodium pentathol in the blood is at least 50 milligrams per kilogram. The half-life of sodium pentathol is 10 hours.

(a) Let *S* represent the amount of sodium pentathol in the bloodstream at time *t*. If sodium pentathol is eliminated from the bloodstream at a rate proportional to the amount present, write the differential equation that governs this process.

(b) Find the general solution to the differential equation in part (a).

(c) Suppose that your friend weighs 100 kg. What single dose of the drug should be given to keep your friend anesthetized for five hours?

**Solution:**

(a) $\boxed{\dfrac{dS}{dt} = -kS, \quad k > 0.}$

(b) $\dfrac{dS}{dt} = -kS \Rightarrow \int \dfrac{dS}{S} = \int -k\, dt \Rightarrow \ln|S| = -kt + \widetilde{C} \Rightarrow |S| = e^{\widetilde{C}} e^{-kt} \Rightarrow \boxed{S(t) = Ce^{-kt}.}$

(c) Half life of 10 hours implies $k = \dfrac{\ln 2}{10}$. $S(t)$ must not drop below 5000 = (50 mg/kg)(100 kg) for 5 hours. This gives $5000 = Ce^{-(\frac{\ln 2}{10} 5)} = Ce^{\ln 2^{-5/10}} = C(2^{-1/2}) = C/\sqrt{2} \Rightarrow C = 5000\sqrt{2}$ mg. Your friend needs $\boxed{\text{a single dose of } 5\sqrt{2} \text{ grams.}}$

**Problem 3:** In this problem, we consider the differential equation

$$\frac{dy}{dt} - \frac{2}{t} y = t^3 \cos t \ .$$

(a) Solve the differential equation using the **Euler-Lagrange** (variation of parameters) method.

(b) Solve the differential equation using the **integrating factor** method.

For full credit, you must show all your work. Do NOT use a formula for the form of the general solution.

**Solution:**

(a) To solve with the Euler-Lagrange method, we first solve the homogeneous problem

$$y' - \frac{2}{t} y = 0 \ ,$$

which can be solved via separation of variables. Upon separating and integrating, we have

$$\int \frac{dy}{y} = \int \frac{2}{t} dt \ \Rightarrow \ \ln|y| = 2\ln|t| + C.$$

Solving for *y*, we find the *homogeneous solution* to be $y_h = C\,t^2$, where *C* is an arbitrary constant.

We next seek a *particular solution* of the form

$$y_p = v(t)t^2.$$

Plugging this in to the differential equation gives $[v'(t)t^2 + 2t\,v(t)] - \frac{2}{t}(v(t)t^2) = t^3 \cos t,$ which simplifies to

$$v'(t) = t\cos(t).$$

Integrating with respect to *t* (using integration by parts), we find

$$v(t) = t\sin t + \cos t.$$

Hence, the particular solution is

$$y_p = v(t)t^2 = t^3 \sin t + t^2 \cos t.$$

The general solution is therefore

$$\boxed{y(t) = y_h(t) + y_p(t) = C\,t^2 + t^3 \sin t + t^2 \cos t \ ,}$$

where *C* is an arbitrary constant.

(b) To solve with the integrating factor method, we first compute the integrating factor

$$\mu(t) = e^{-\int \frac{2}{t} dt} = e^{-2\ln|t|} = \frac{1}{t^2}.$$

Multiplying the differential equation by $\mu(t)$, we obtain

$$\frac{1}{t^2} y' - \frac{2}{t^3} y = t\cos t.$$

Rewriting the left hand side via the product rule, we have $\frac{d}{dt}\left[\frac{1}{t^2} y\right] = t\cos t.$ We next integrate both sides of the equation with respect to $t$ to get

$$\int \frac{d}{dt}\left[\frac{1}{t^2} y\right] dt = \int t\cos t\, dt \quad \Rightarrow \quad \frac{1}{t^2} y = t\sin t + \cos t + C$$

and thus (as in part (a)) the general solution

$$\boxed{y = t^3 \sin t + t^2 \cos t + C t^2.}$$

**Problem 4:** Consider a logistic growth model of a population $y(t)$ with constant rate of harvesting $h > 0$,

$$y'(t) = k\, y\left(1 - \frac{y}{L}\right) - h,$$

where $k > 0$ is the intrinsic growth rate and $L > 0$ is the carrying capacity. The parameters $k$ and $L$ take the constant values $k = 2$ and $L = 1$.

(a) For which values of $h$ does the system have two equilibria? Find then their values $y_1(h)$ and $y_2(h)$.

(b) In the case of two equilibria, determine their stability.

(c) If the two equilibria coincide, what is the stability of the resulting single equilibrium?

**Solution:**

(a) We obtain the equilibria by setting the ODE's RHS to zero and solving the resulting quadratic equation for $y$. With the given values: $2y(1-y) - h = 0$, i.e. $y^2 - y + \frac{1}{2}h = 0$, with the solutions $\boxed{y_{1,2} = \frac{1}{2}(1 \pm \sqrt{1-2h}).}$ $\boxed{\text{This provides two solutions when } h < \frac{1}{2}.}$

(b) The ODE RHS can be written in terms of the two roots as $-2(y-y_1)(y-y_2)$, i.e. the ODE becomes

$$y' = -2(y-y_1)(y-y_2).$$

Say that we have ordered the roots such that $y_1 < y_2$. The signs of the derivative $y'$ are then:

$$0 \le y < y_1 \quad : y' < 0, \qquad y_1 < y < y_2 \quad : y' > 0, \qquad y > y_2 \quad : y' < 0.$$

Therefore, $\boxed{y = y_1 \text{ is unstable, and } y = y_2 \text{ is stable.}}$

(c) The two equilibria coincide when $y_1 = y_2 = \frac{1}{2}$ in which case the ODE becomes $y' = -2(y - \frac{1}{2})^2$. This is negative unless $y = \frac{1}{2}$, so $\boxed{\text{this is a semi-stable equilibrium.}}$

**Problem 5:** A 10 gallon water tank starts out empty at time $t = 0$. The inflow rate from that moment on is 1 gallon / second, and this inflow contains a (decreasing) amount of $\dfrac{1}{1+t}$ oz. of salt per gallon. The tank is old and it leaks badly, losing ½ gallon per second.

(a) At what time $t = T$ is the tank full?

(b) Let $x(t)$ denote the total amount of salt in the tank (assumed to be well stirred). Write down the ODE that describes $x(t)$ for $0 \le t \le T$, and give the initial condition for $t = 0$.

(c) Find an analytic expression for the function $x(t)$ (again for $0 \le t \le T$).

**Solution:**

(a) For water, {rate in} – {rate out} = $1 - \frac{1}{2} = \frac{1}{2}$ gallons/second, so the 10 gallon tank is full at time $\boxed{T = 20.}$

(b) The governing equation for the amount of an added chemical in the tank takes the form

$$\underset{\substack{\uparrow \\ \text{change} \\ \text{in tank}}}{\frac{dx}{dt}} = \underset{\substack{\uparrow \\ (\text{concentration in})\times \\ (\text{flow rate in})}}{\begin{Bmatrix}\text{rate}\\ \text{in}\end{Bmatrix}} - \underset{\substack{\uparrow \\ (\text{concentration out})\times \\ (\text{flow rate out})}}{\begin{Bmatrix}\text{rate}\\ \text{out}\end{Bmatrix}}$$

With the numbers for the present problem, this equation becomes: $\dfrac{dx}{dt} = \dfrac{1}{1+t} \cdot 1 - \dfrac{x}{t/2} \cdot \dfrac{1}{2}$, or slightly simpler, $\boxed{x' + \dfrac{x}{t} = \dfrac{1}{1+t}}$. Since there is no salt at $t = 0$, the initial condition is $\boxed{x(0) = 0.}$

(c) Having written the ODE in the form $x' + \dfrac{x}{t} = \dfrac{1}{1+t}$, its *integrating factor* becomes $e^{\int \frac{1}{t} dt} = e^{\ln t} = t$.

Multiplying with this factor $t$ gives as LHS (left hand side) $tx' + x = \dfrac{d}{dt}(t\,x)$ and as RHS is

$\dfrac{t}{1+t} = \dfrac{t+1-1}{1+t} = 1 - \dfrac{1}{1+t} = \dfrac{d}{dt}(t - \ln(1+t))$. Integrating both sides and then dividing by $t$ gives $x(t) = 1 - \dfrac{\ln(1+t)}{t} + \dfrac{C}{t}$. Since $\dfrac{\ln(1+t)}{t} \to 1$ when $t \to 0$, the initial condition $x(0) = 0$ tells that $C = 0$, and therefore $\boxed{x(t) = 1 - \dfrac{\ln(1+t)}{t} \quad \text{(oz.), valid for } 0 \le t \le 20.}$