Each problem worth 15 points, except problem 1 worth 10 points

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Problem 1 (True/False) Are the following statements true or false? (Only answer TRUE or FALSE is needed; no justification is required.) If a statement is not always true, reply 'FALSE'. Give your answers in the bluebook (and not on this problem sheet).

(a) The solutions to $\frac{dy}{dt} = -\frac{t}{y}$ are all circles around the origin in the (t, y) plane.

(b) The ODE $y' = \sqrt{y}$, with $y(0) = 0$, has $y(t) = 0$ as its only solution.

(c) The system $\begin{cases} x' = x(2-y) \\ y' = y(-3 + \frac{1}{2}x) \end{cases}$ has four equilibrium points.

(d) The solution space to $y''' = y$ is spanned by $\{e^t, e^{-t/2}\cos(\frac{\sqrt{3}}{2}t), e^{-t/2}\sin(\frac{\sqrt{3}}{2}t)\}$.

(e) The eigenvalues of a 2×2 matrix A depend only on $\text{Tr}(A)$ and $\det(A)$ (with Tr and det standing for trace and determinant, respectively).

(f) If the Wronskian for two functions is identically zero, the two functions are linearly dependent.

(g) A solution to the ODE $t^2y'' - 3ty' + 3y = 0$ is $y(t) = t^3$.

(h) If A is a 3×3 matrix, then $\det(2A) = 2\det(A)$

(i) If $A\vec{x} = \vec{b}$ and A is a square matrix, then there is always a solution $\vec{x}$.

(j) The magnitude of $y(t)$ is bounded over all time t if y solves $y'' + y = \cos(t)$.

Solution:

(a) True

(b) False, the uniqueness part of Picard's theorem fails since $d/dy f(y)$ is not continuous at 0. See Example 3 in section 1.5 (and problem 21)

(c) False, it has two equilibrium points

(d) True, which you can verify by showing that 1 and $-\frac{1}{2} \pm \frac{\sqrt{3}}{2} \cdot i$ are roots of the equation $r^3 - 1 = 0$. Alternatively, verify that each of these 3 functions solves the equation, and use the Wronskian to show these 3 functions are linearly independent.

(e) True

(f) False, the test is inconclusive then

(g) True. This is an Euler-Cauchy equation, and even if you can't solve it, you can easily verify if a given function is a solution or not.

(h) False, it is $|2A| = 2^3|A|$ when A is 3×3

(i) False

(j) False, this is resonance

Problem 2 Find the general solution to the ODE $y'(t) + 3t^2 \cdot y(t) = t^2$ by means of:

(a) the integrating factor method,

(b) solving the homogeneous problem for $y_h(t)$ and then obtaining a particular solution $y_p(t)$ by carrying out the method of variation of parameters.

Solution:

(a) The integrating factor is $\mu(t) = e^{\int 3t^2\,dt} = e^{t^3}$. Multiplying the ODE with this factor gives

$$\frac{d}{dt}\left(e^{t^3} \cdot y(t)\right) = e^{t^3} \cdot t^2.$$

Integrating both sides gives

$$\begin{aligned} e^{t^3} \cdot y(t) &= \int e^{t^3} t^2\,dt \\ &= \frac{1}{3}\int e^u\,du \quad (u\text{-sub with}\, u = t^3) \\ &= \frac{1}{3}e^u + C \\ &= \frac{1}{3}e^{t^3} + C \end{aligned}$$

hence, after dividing by μ, $\boxed{y(t) = \frac{1}{3} + C \cdot e^{-t^3}}$.

(b) First, find y_h which solves $y_h' + 3t^2 y_h = 0$, which is separable, so write it as $y_h'/y_h = -3t^2$ and integrate both sides to get $\ln|y_h(t)| = -t^3 + c$, and then exponentiate both sides to get $\boxed{y_h(t) = C \cdot e^{-t^3}}$ where C is a real constant.

Then try $y_p(t) = v(t) \cdot e^{-t^3}$ so $y_p' = v' \cdot e^{-t^3} + v \cdot -3t^2 \cdot e^{-t^3}$. Plugging this into the full ODE, $v \cdot y_p' + v \cdot 3t^2 \cdot y_p$ cancel, leaving $v' \cdot e^{-t^3} = t^2$, hence

$$\begin{aligned} v(t) = \int v'(t)\,dt &= \int t^2 e^{t^3}\,dt \\ &= \frac{1}{3}e^{t^3} \end{aligned}$$

(this is the same integral as in part (a)), and note we can leave off the constant of integration. Thus $y_p(t) = v(t)\cdot e^{-t^3} = \frac{1}{3}e^{t^3}\cdot e^{-t^3} = \frac{1}{3}$, so altogether, the solution is $\boxed{y(t) = y_p + y_h = \frac{1}{3} + C \cdot e^{-t^3}}$.

Problem 3 The following parts (a) and (b) are unrelated.

(a) Consider the system

$$\begin{aligned} dR/dt &= R\,(3 - R - 2S) \\ dS/dt &= S\,(2 - S - R) \end{aligned}$$

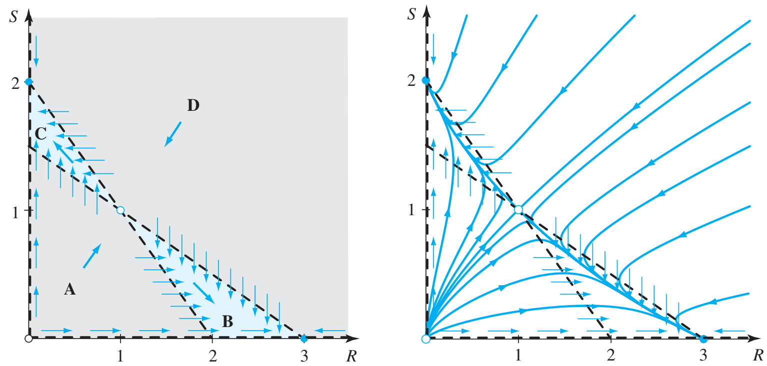
which could be used to describe the population of rabbits $R(t)$ and sheep $S(t)$ in a competition-for-resources model, for example. If $R(0) = 1.1$ and $S(0) = 0.9$, what is the limiting behavior of $R(t)$ and $S(t)$ as $t \to \infty$? [It is more important to describe your reasoning than it is to get the right answer; you may find it useful to draw nullclines]

(b) Suppose coffee with a concentration of 200 g/L of caffeine per liter is poured at a rate of 2 liters/minute into a container that initially contains 5 liter of pure water, and simultaneously, the container is drained at a rate of 3 liters/minute (assume the liquid is always well-mixed). Write-down a mathematical model for this system *and* describe what techniques you could use to solve the model (no need to actually solve it).

Solution:

(a) See Example 4 in section 2.6 in the book; there are 4 equilibrium solutions, $\{(0,0),(0,2),(3,0),(1,1)\}$, and by looking at the nullclines in the phase plane [see below], we have an indication that $\boxed{\{(0,0),(1,1)\} \text{ are } \textit{unstable equilibria}}$ and that $\boxed{\{(0,2),(3,0)\} \text{ are } \textit{asymptotically stable equilibria}}$. Since the initial point is somewhat close to $(3,0)$, we guess that it will be attracted to that equilibrium, so the answer is $\boxed{(3,0)}$.

Nullclines (left) and some sample trajectories (right) are shown below:



(b) Let $y(t)$ be the amount of caffeine in the container (in grams) at time t (in minutes). Then we model

$$dy/dt = 2\,\text{L/min} \cdot 200\,\text{g/L} - 3\,\text{L/min} \cdot \frac{y(t)\,\text{g}}{(5-t)\,\text{L}}, \qquad y(0) = 0.$$

Problem 4 Consider

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

(a) Find the determinant of A.

(b) Does $A\vec{x} = \vec{b}$ have a unique solution? Give a brief justification.

(c) Solve for $\vec{x}$ if $A\vec{x} = \vec{b}$.

(d) Does the set $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \right\}$ form a basis for $\mathbb{R}^3$? Give a brief justification.

Solution:

(a) $|A| = (1)\,[(2)(-1)-(1)(0)] - (-1)\,[(2)(-1)-(3)(1)] + (0)\,[(2)(0)-(2)(3)] = -7$

(b) A is nonsingular, since $|A| \neq 0$, meaning that $A\vec{x} = \vec{b}$ has a unique solution.

(c)

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ -1 & 2 & 0 & 2 \\ 0 & 1 & -1 & -1 \end{array}\right] \overset{R_2 = R_2 + R_1}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 4 & 3 & 3 \\ 0 & 1 & -1 & -1 \end{array}\right] \overset{R_1 = R_1 - 2R_3}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 3 \\ 0 & 4 & 3 & 3 \\ 0 & 1 & -1 & -1 \end{array}\right]$$

$$\overset{R_3 = -\frac{4}{7}\left(R_3 - \frac{1}{4}R_2\right)}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 3 \\ 0 & 4 & 3 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \overset{\substack{R_2 = \frac{R_2 - 3R_2}{4} \\ R_1 = R_1 - 5R_3}}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Thus, $\vec{x} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

(d) Yes. All three belong to $\mathbb{R}^3$ and are linearly independent since they are the column vectors of A and $|A| \neq 0$.

Problem 5 The following parts (a) and (b) are unrelated.

(a) Find the periodic steady state of the harmonic oscillator with mass $m = 1$, friction $b = 2$ and spring constant $k = 9$ under the action of external force $F(t) = 2\sin(3t)$.

(b) Find the general solution $y(t)$ of the equation $y'' - 2y' + y = \frac{e^t}{t+1}$.

Solution:

(a) The equation of motion for the oscillator position $x(t)$ is

$$x'' + 2x' + 9x = 2\sin(3t).$$

The characteristic roots are

$$r = -1 \pm \frac{\sqrt{4-36}}{2} = -1 \pm 2\sqrt{2}i,$$

their real part is negative, therefore the general homogeneous solution goes to zero as $t \to \infty$ (this is always true if $m, b, k > 0$). The particular inhomogeneous solution can be found in the form

$$x_p(t) = A\cos(3t) + B\sin(3t).$$

Differentiating, we find

$$x_p'(t) = -3A\sin(3t) + 3B\cos(3t), \qquad x_p''(t) = -9(A\cos(3t) + B\sin(3t)) = -9x_p.$$

Substitution of x_p and its derivatives into the ODE gives

$$2(-3A\sin(3t) + 3B\cos(3t)) = 2\sin(3t),$$

i.e. we find $A = -1/3$ and $B = 0$. Thus, the particular solution is

$$x_p(t) = -\frac{1}{3}\cos(3t).$$

As $t \to \infty$, it persists, therefore the periodic steady state is given by these oscillations, $x_{st}(t) = x_p(t) = -\frac{1}{3}\cos(3t)$.

(b) The characteristic equation is

$$r^2 - 2r + 1 = (r-1)^2 = 0,$$

so we have double root $r = 1$. The general homogeneous solution is then

$$y_h(t) = (C_1 t + C_2)e^t,$$

with two arbitrary constants C_1 and C_2. Using the method of variation of parameters, we look for a particular solution y_p in the form

$$y_p(t) = (tv_1(t) + v_2(t))e^t.$$

The two linearly independent homogeneous solutions are $y_1 = te^t$ and $y_2 = e^t$ and their Wronskian is

$$W = \begin{vmatrix} te^t & e^t \\ (t+1)e^t & e^t \end{vmatrix} = te^{2t} - (t+1)e^{2t} = -e^{2t}.$$

We find v_1' and v_2' as (the Cramer rule)

$$v_1' = -e^{-2t}\begin{vmatrix} 0 & e^t \\ \frac{e^t}{t+1} & e^t \end{vmatrix} = \frac{1}{t+1}, \qquad v_2' = -e^{-2t}\begin{vmatrix} te^t & 0 \\ (t+1)e^t & \frac{e^t}{t+1} \end{vmatrix} = -\frac{t}{t+1}.$$

Integrating the last expressions, we get

$$v_1 = \ln|t+1|, \qquad v_2 = \int\left(\frac{1}{t+1} - 1\right)dt = \ln|t+1| - t.$$

(Another way to see the v_2 integral: make a u-sub with $u = t+1$). Thus, the general solution is

$$y(t) = y_h + y_p = (C_1 t + C_2)e^t + (t\ln|t+1| + \ln|t+1| - t)e^t.$$

Problem 6

(a) Find the Laplace transform $\mathcal{L}\{y(t)\}$ of the function $y(t) = \begin{cases} t^2 & 0 \le t < 1 \\ 0 & t \ge 1. \end{cases}$

(b) Find $\mathcal{L}\{y'''(t)\}$ where $y(t) = t^3 e^{-t}$ (simplify as much as possible).

(c) Find $y(t)$ if we are given that $\mathcal{L}\{y(t)\} = \frac{s \cdot e^{-2s}}{s^2-1}$.

Solution:

(a) While you could directly integrate $Y(s) = \int_0^1 t^2 e^{-st}\,dt$ using integration by parts twice, it's simpler to note that $y(t) = t^2 \cdot (1 - \text{step}(t-1))$ (or $y(t) = t^2 \cdot (\text{step}(t) - \text{step}(t-1))$ is valid as well). Then since t^2 corresponds to multiplication in the Laplace domain, this means

$$\begin{aligned} Y(s) = (-1)^2 \frac{d^2}{ds^2} L\{1 - \text{step}(t-1)\} &= \frac{d^2}{ds^2}\left(\frac{1}{s}\left(1 - e^{-s}\right)\right) \\ &= \frac{d}{ds}\left(\frac{e^{-s}s - (1 - e^{-s}}{s^2}\right) \\ &= \frac{(-e^s s + e^{-s} - e^{-s})s^2 - 2s(e^{-s}s - (1 - e^{-s}))}{s^4} \\ &= \boxed{\frac{-e^{-s} \cdot \left(s^2 + 2s + 2\right) + 2}{s^3}}. \end{aligned}$$

using the quotient rule twice.

If you do integrate by parts twice, it looks like

$$\begin{aligned} Y(s) = \int_0^1 t^2 e^{-st}\,dt &= \frac{-1}{s} e^{-st} t^2 |_0^1 - \frac{-2}{s} \int_0^1 t e^{-st}\,dt \\ &= \frac{-e^{-s}}{s} + \frac{2}{s}\left(\frac{-1}{s} e^{-st} t|_0^1 - \frac{-1}{s}\int_0^1 e^{-st}\,dt\right) \\ &= \frac{-e^{-s}}{s} + \frac{2}{s}\left(\frac{-e^{-s}}{s} - \frac{-1}{s}\frac{-1}{s} e^{-st}|_0^1\right) \\ &= \frac{-e^{-s}}{s} + \frac{2}{s}\left(\frac{-e^{-s}}{s} - \frac{1}{s^2}(e^{-s} - 1)\right) \\ &= \frac{-e^{-s}\left(s^2 + 2s + 2\right) + 2}{s^3} \end{aligned}$$

(b) You could differentiate three times and then take the Laplace transform and simplfy, but it's simpler to use the formula $\mathcal{L}\{y'''(t)\} = s^3 Y(s) - s^2 y(0) - s y'(0) - y''(0)$ where $Y(s) = L\{y(t)\}$. From using the table of Laplace transforms, $Y(s) = \frac{6}{(s+1)^4}$. Since $y(0) = y'(0) = y''(0) = 0$, the answer is $\boxed{L\{y'''(t)\} = \frac{6s^3}{(s+1)^4}}$.

If you differentiate, then

$$\begin{aligned} y(t) &= t^3 e^{-t} \\ y'(t) &= 3t^2 e^{-t} - t^3 e^{-t} \\ y''(t) &= 6te^{-t} - 3t^2 e^{-t} - 3t^2 e^{-t} + t^3 e^{-t} = 6te^{-t} - 6t^2 e^{-t} + t^3 e^{-t} \\ y'''(t) &= 6e^{-t} - 6te^{-t} - 12te^{-t} + 6t^2 e^{-t} + 3t^2 e^{-t} - t^3 e^{-t} \\ &= e^{-t}\left(-t^3 + 9t^2 - 18t + 6\right) \end{aligned}$$

so

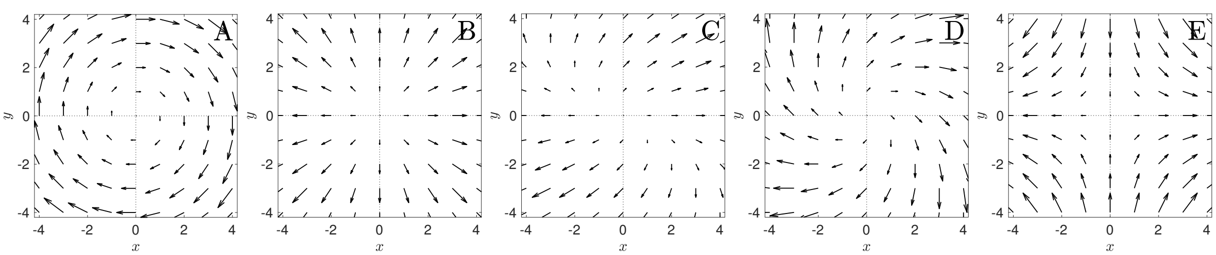
$$
\begin{aligned}
Y(s) &= -\frac{3!}{(s+1)^4} + 9\frac{2!}{(s+1)^3} - 18\frac{1!}{(s+1)^2} + 6\frac{0!}{(s+1)} \\
&= \frac{-6 + 18(s+1) - 18(s+1)^2 + 6(s+1)^3}{(s+1)^4} \\
&= \frac{-6 + 18s + 18 - 18(s^2+2s+1) + 6(s^3+3s^2+3s+1)}{(s+1)^4} = \frac{6s^3}{(s+1)^4}
\end{aligned}
$$

(c) By using partial fractions, write $Y(s) = \frac{se^{-2s}}{s^2-1} = e^{-2s}\left(\frac{1/2}{s+1} + \frac{1/2}{s-1}\right)$. The form now matches the right-hand-side of the delayed function formula $L\{f(t-c)\text{step}(t-c)\} = e^{-cs}L\{f(t)\}$, with $f(t) = \frac{1}{2}e^{-t} + \frac{1}{2}e^{t} = \cosh(t)$, hence $\boxed{y(t) = \cosh(t-2)\cdot\text{step}(t-2)}$.

Problem 7 For both problems, let $\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.

(a) For $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, solve $\frac{d}{dt}\vec{x} = A\vec{x}$ with $\vec{x}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

(b) If $\frac{d}{dt}\vec{x} = A\vec{x}$, match choices of the matrix A with corresponding phase plane portraits:
(1) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, (2) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, (3) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, (4) $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, (5) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$,



Solution:

(a) Solve for the eigenvalues (using the quadratic formula) to find $\lambda = 1 \pm i$ (i.e., $\alpha = 1$ and $\beta = 1$). For $\lambda = 1 + i$, with corresponding eigenvector $\vec{v} = (v_1, v_2)$, we find $-iv_1 + v_2 = 0$, and choose any value of v_2 that we like so long as $\vec{v} \neq 0$. A simple choice is $v_2 = i$ so that $v_1 = 1$, hence $\vec{v} = (1, i)$.

Writing $\vec{v} = \vec{p} + i\vec{q}$ for real vectors $\vec{p}$ and $\vec{q}$ gives $\vec{p} = (1,0)$ and $\vec{q} = (0,1)$. Now, use the formula for complex eigenvalues:

$$
\begin{aligned}
\vec{x}(t) &= c_1 e^{\alpha t}\left(\cos(\beta t)\vec{p} - \sin(\beta t)\vec{q}\right) + c_2 e^{\alpha t}\left(\sin(\beta t)\vec{p} + \cos(\beta t)\vec{q}\right) \\
&= c_1 e^{t}\left(\cos(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + c_2 e^{t}\left(\sin(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)
\end{aligned}
$$

so at $t = 0$,

$$
\begin{aligned}
\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \vec{x}(0) &= c_1 e^{t}\left(\cos(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + c_2 e^{t}\left(\sin(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\Bigg|_{t=0} \\
&= c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

which we solve to find $c_1 = 3, c_2 = 0$, hence our answer is $\boxed{\vec{x}(t) = 3e^{t}\left(\cos(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)}$.

(b) $1 = \text{A}$, $2 = \text{E}$, $3 = \text{B}$, $4 = \text{D}$, $5 = \text{C}$