APPM 2360: Final Exam

December 17, 2018. 2.5 hours.

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your instructor's name, (3) your lecture section number and (4) a grading table. Text books, class notes, cell phones and calculators are NOT permitted. A one page (letter sized **2 sided**) crib sheet is allowed. You must work all of the problems on the exam. Unless indicated, show ALL of your work in your bluebook and $\boxed{\text{box}}$ in your final answer. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit.

Problem 1 (True/False) Are the following statements true or false? (Only answer TRUE or FALSE is needed; no justification is required.) If a statement is not always true, reply 'FALSE'. Give your answers in the bluebook (and not on this problem sheet).

(a) The solutions to $\frac{dy}{dt} = -\frac{t}{y}$ are all circles around the origin in the (t, y) plane.

(b) The ODE $y' = \sqrt{y}$, with $y(0) = 0$, has $y(t) = 0$ as its only solution.

(c) The system $\begin{cases} x' = x(2-y) \\ y' = y(-3 + \frac{1}{2}x) \end{cases}$ has four equilibrium points.

(d) The solution space to $y''' = y$ is spanned by $\{e^t, e^{-t/2}\cos(\frac{\sqrt{3}}{2}t), e^{-t/2}\sin(\frac{\sqrt{3}}{2}t)\}$.

(e) The eigenvalues of a 2×2 matrix A depend only on $\mathrm{Tr}(A)$ and $\det(A)$ (with Tr and det standing for trace and determinant, respectively).

(f) If the Wronskian for two functions is identically zero, the two functions are linearly dependent.

(g) A solution to the ODE $t^2y'' - 3ty' + 3y = 0$ is $y(t) = t^3$.

(h) If A is a 3×3 matrix, then $\det(2A) = 2\det(A)$

(i) If $A\vec{x} = \vec{b}$ and A is a square matrix, then there is always a solution $\vec{x}$.

(j) The magnitude of $y(t)$ is bounded over all time t if y solves $y'' + y = \cos(t)$.

Problem 2 Find the general solution to the ODE $y'(t) + 3t^2 \cdot y(t) = t^2$ by means of:

(a) the integrating factor method,

(b) solving the homogeneous problem for $y_h(t)$ and then obtaining a particular solution $y_p(t)$ by carrying out the method of variation of parameters.

Problem 3 The following parts (a) and (b) are unrelated.

(a) Consider the system

$$dR/dt = R\,(3 - R - 2S)$$
$$dS/dt = S\,(2 - S - R)$$

which could be used to describe the population of rabbits $R(t)$ and sheep $S(t)$ in a competition-for-resources model, for example. If $R(0) = 1.1$ and $S(0) = 0.9$, what is the limiting behavior of $R(t)$ and $S(t)$ as $t \to \infty$? [It is more important to describe your reasoning than it is to get the right answer; you may find it useful to draw nullclines]

(b) Suppose coffee with a concentration of 200 g/L of caffeine per liter is poured at a rate of 2 liters/minute into a container that initially contains 5 liter of pure water, and simultaneously, the container is drained at a rate of 3 liters/minute (assume the liquid is always well-mixed). Write-down a mathematical model for this system *and* describe what techniques you could use to solve the model (no need to actually solve it).

Problem 4 Consider

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

(a) Find the determinant of A.

(b) Does $A\vec{x} = \vec{b}$ have a unique solution? Give a brief justification.

(c) Solve for $\vec{x}$ if $A\vec{x} = \vec{b}$.

(d) Does the set $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \right\}$ form a basis for $\mathbb{R}^3$? Give a brief justification.

Problem 5 The following parts (a) and (b) are unrelated.

(a) Find the periodic steady state of the harmonic oscillator with mass $m = 1$, friction $b = 2$ and spring constant $k = 9$ under the action of external force $F(t) = 2\sin(3t)$.

(b) Find the general solution $y(t)$ of the equation $y'' - 2y' + y = \frac{e^t}{t+1}$.

Problem 6

(a) Find the Laplace transform $\mathcal{L}\{y(t)\}$ of the function $y(t) = \begin{cases} t^2 & 0 \le t < 1 \\ 0 & t \ge 1. \end{cases}$

(b) Find $\mathcal{L}\{y'''(t)\}$ where $y(t) = t^3 e^{-t}$ (simplify as much as possible).

(c) Find $y(t)$ if we are given that $\mathcal{L}\{y(t)\} = \frac{s \cdot e^{-2s}}{s^2 - 1}$.

Problem 7 For both problems, let $\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.

(a) For $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, solve $\frac{d}{dt}\vec{x} = A\vec{x}$ with $\vec{x}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

(b) If $\frac{d}{dt}\vec{x} = A\vec{x}$, match choices of the matrix A with corresponding phase plane portraits:

(1) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, (2) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, (3) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, (4) $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, (5) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$,

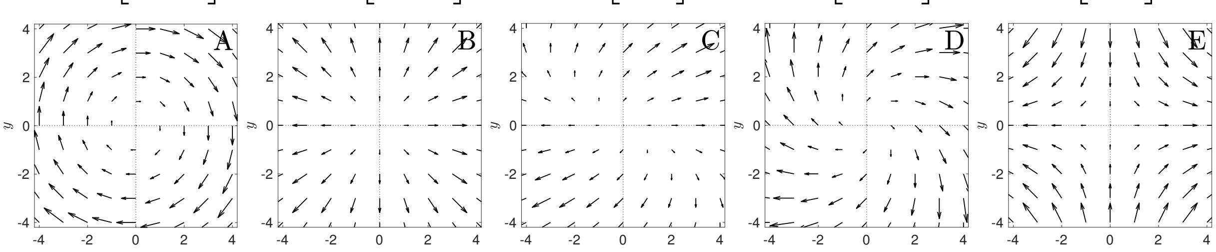

Table of Laplace Transforms. $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$

$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s)$	$\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$	$\mathcal{L}\{\delta(t)\} = 1$
$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$	$\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$	$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$
$\mathcal{L}\{\text{step}(t)\} = \frac{1}{s}$	$\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$	$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$
$\mathcal{L}\{f(t-a)\text{step}(t-a)\} = e^{-as}F(s)$	$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$	