Problem 1 (True/False) Are the following statements true or false? (Only answer TRUE or FALSE is needed; no justification is required.) If a statement is not always true, reply 'FALSE'. Give your answers in the bluebook (and not on this problem sheet).

(a) Observe that $y(t) = \cos(2t)$ is a solution to the ODE $y'' - y' + 4y = 2\sin(2t)$. Is $y(t) = -\cos(2t)$ also a solution to this ODE?

(b) When using the method of undetermined coefficients for the ODE $y'' + y = 6\cos(t)$ the particular solution should be of the form $y_p(t) = A\cos(t) + B\sin(t)$.

(c) $-2\mathcal{L}\{te^{-t^2}\} = s\mathcal{L}\{e^{-t^2}\} - 1$, where $\mathcal{L}\{f(t)\}$ is the Laplace transform of $f(t)$.

(d) The functions t^3 and $|t^3|$ have the same Laplace transform.

(e) The Laplace transform $\mathcal{L}\{e^{t^3}\}$ exists for some s.

(f) All solutions to the equation $y'' + 4y = 0$ can be written in the form $y = A\sin(2t + \delta)$ for some real numbers A and δ.

(g) All solutions to the equation $y'' + 2y' + y = 0$, can be written in the form $y = ce^{-t}$ for some real number c.

(h) The equation $y''' + 2y'' + 4t^2y' - y = \cos(t)e^{-t^2}$ can be converted to 3 first-order linear ODEs in 3 dependent variables.

(i) The set of functions $\{\cos(t), \cos(t+1), \cos(t+2)\}$ is linearly independent.

(j) The set of functions $\{\cos(t), \cos(2t), \cos(3t)\}$ is linearly independent.

Solution:

(a) FALSE

(b) FALSE

(c) TRUE

(d) TRUE

(e) FALSE

(f) TRUE

(g) FALSE

(h) TRUE

(i) FALSE

(j) TRUE

Problem 2 The following questions are unrelated.

(a) Find a particular solution to

$$y'' - 4y' - 12y = 4e^{3t}$$

using the method of undetermined coefficients.

(b) Find the <u>full</u> general solution to

$$y'' + 4y = \frac{1}{\cos(2t)}.$$

<u>Hint:</u> $\int \tan(at)dt = -\frac{1}{a}\ln(\cos(at)) + C$

Solution:

(a) First we guess $y_p(t) = Ae^{3t}$. We plug this into our DE,

$$\left(Ae^{3t}\right)'' - 4\left(Ae^{3t}\right)' - 12\left(Ae^{3t}\right) = 9Ae^{3t} - 12Ae^{3t} - 12Ae^{3t} = -15Ae^{3t} = 4e^{3t}$$

which gives us that $-15A = 4$ or that $A = -\frac{4}{15}$ and $\boxed{y_p(t) = -\frac{4}{15}e^{3t}}$.

(b) First start by finding the homogeneous solution,

$$y_h'' + 4y_h = 0$$

with characteristic equation $r^2 + 4 = 0$ giving $r = \pm 2i$. Thus,

$$y_h(t) = C_1\cos(2t) + C_2\sin(2t)$$

To find the particular solution we need to use variation of parameters. Set $y_1 = \cos(2t)$ and $y_2 = \sin(2t)$, then $y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$. The Wronskian is,

$$W[\cos(2t), \sin(2t)] = \begin{vmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{vmatrix} = 2\cos^2(2t) + 2\sin^2(2t) = 2$$

Now,

$$v_1' = \frac{-\sin(2t)}{2\cos(2t)} = -\frac{1}{2}\tan(2t)$$

$$\Rightarrow v_1 = -\frac{1}{2}\int \tan(2t)dt = \frac{1}{4}\ln(\cos(2t))$$

and

$$v_2' = \frac{\cos(2t)}{2\cos(2t)} = \frac{1}{2}$$

$$\Rightarrow v_2 = \int \frac{1}{2}dt = \frac{t}{2}$$

Then,

$$y_p(t) = \frac{1}{4}\ln(\cos(2t))\cos(2t) + \frac{1}{2}t\sin(2t)$$

Finally,

$$\boxed{y(t) = C_1\cos(2t) + C_2\sin(2t) + \frac{1}{4}\ln(\cos(2t))\cos(2t) + \frac{1}{2}t\sin(2t)}$$

Problem 3

(a) Find a solution to the initial value problem $y'' + 4y' + 13y = 0$, $y(0) = 1$, $y'(0) = 10$.

Solution: Trying $y = e^{rt}$, the characteristic equation is $r^2 + 4r + 13 = 0$. Applying the quadratic formula, the roots are $r = \frac{-4 \pm \sqrt{16-52}}{2} = -2 \pm 3i$.

From here, one could proceed by writing $y = \tilde{c}_1 e^{(-2+3i)t} + \tilde{c}_2 e^{(-2-3i)t}$ and solve for the complex coefficients $\tilde{c}_1$ and $\tilde{c}_2$ but this is more work.

A better approach is to write $y = e^{-2t}(c_1 \cos(3t) + c_2 \sin(3t))$. Then $y(0) = e^0(c_1)$, so the initial condition $y(0) = 1$ implies $c_1 = 1$.

Then, using the product rule to take the derivative of $y = e^{-2t}(\cos(3t) + c_2 \sin(3t))$ gives $y' = -2e^{-2t}(\cos(3t) + c_2 \sin(3t)) + e^{-2t}(-3\sin(3t) + c_2 3\cos(3t))$. Hence $y'(0) = -2 + 3c_2$, so applying the initial condition $y'(0) = 10$ means $10 = -2 + 3c_2$, so $c_2 = 4$.

Overall, the solution is $\boxed{y(t) = e^{-2t}(\cos(3t) + 4\sin(3t))}$.

(b) Find a basis for the space of solutions to the equation $y''' - y'' = 0$.

Solution: Trying $y = e^{rt}$, the characteristic equation is $r^3 - r^2 = 0$, which factors as $r^2(r-1) = 0$ so $r = 0$ is a double-root and $r = 1$ is a single root. A basis is therefore $\boxed{\{1, t, e^t\}}$. Note than $c_1 + c_2 t + c_3 e^t$ is not a valid answer — this is not the right form of a basis.

(c) Match each ODE with a graph that could correspond to a solution to the ODE

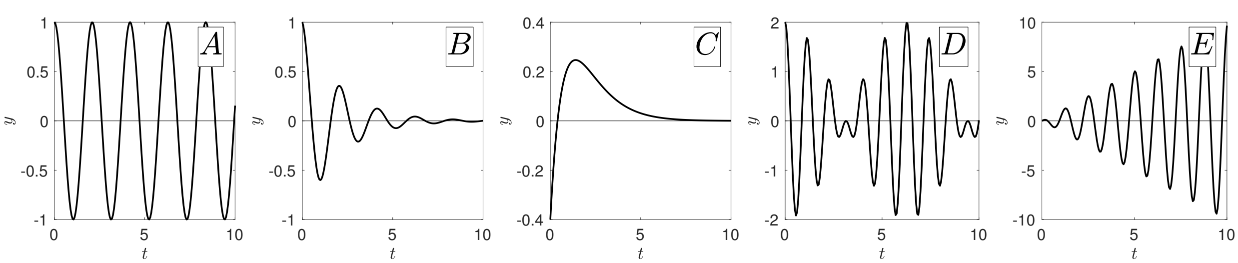
(i) $y'' + 2y' + y = 0$

(ii) $y'' + 25y = -10\sin(5t)$

(iii) $y'' + 9y = 0$

(iv) $y'' + 25y = 11\cos(6t)$

(v) $y'' + y' + 9.25y = 0$



Solution:

(i) C (this graph shows critical or over-damping behavior)

(ii) E (this graph shows resonance)

(iii) A (this graph shows periodic motion with constant amplitude)

(iv) D (this graph shows beats)

(v) B (this graph shows under-damping behavior)

Problem 4 Solve the initial value problem

$$y'' - 5y' + 6y = 0$$

with $y(0) = 2$ and $y'(0) = 2$ by using Laplace transforms.
Solution: First observe,

$$\begin{aligned}\mathcal{L}\{y''\} &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s - 2 \\ \mathcal{L}\{y'\} &= sY(s) - y(0) = sY(s) - 2 \\ \mathcal{L}\{y\} &= Y(s)\end{aligned}$$

Then our ODE becomes,

$$\left(s^2 - 5s + 6\right)Y(s) = 2s - 8$$

Solving for $Y(s)$,

$$Y(s) = \frac{2s-8}{s^2-5s+6} = \frac{2s-8}{(s-2)(s-3)}$$

The partial fraction decomposition should take the form $\frac{A}{s-2} + \frac{B}{s-3}$ leading to $A + B = 2$ and $3A + 2B = 8$. This gives us that $A = 4$ and $B = -2$. Therefore,

$$Y(s) = \frac{4}{s-2} - \frac{2}{s-3}$$

Notice that $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$. This means that,

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{\mathcal{L}\{4e^{2t} - 2e^{3t}\}\} = \boxed{4e^{2t} - 2e^{3t}}$$

Problem 5:

(a) Find the Laplace transform of the function $f(t)$,

$$f(t) = \begin{cases} 2 & 0 \le t < 1, \\ 8e^{-t} & t \ge 1. \end{cases}$$

(b) Find the inverse Laplace transform of the function

$$F(s) = \frac{-7}{(s-1)^2+4} + \frac{8}{(s-1)^2+5}$$

Solution:

(a)

$$\begin{aligned}F(s) &= \int_0^\infty e^{-st}f(t)dt \\ &= 2\int_0^1 e^{-st}dt + 8\int_1^\infty e^{-st}e^{-t}dt \\ &= \boxed{\frac{2}{s}\left(1 - e^{-s}\right) + \frac{8}{s+1}e^{-(s+1)}}\end{aligned}$$

Alternative Solution:

$$f(t) = 2 + (8e^{-t} - 2)\text{step}(t-1)$$
$$\Longrightarrow$$
$$\begin{aligned} F(s) &= \mathcal{L}\{2\} + \mathcal{L}\{(8e^{-t} - 2)\text{step}(t-1)\} \\ &= \left[\frac{2}{s}\right] + \left[e^{-s}\mathcal{L}\{8e^{-(t+1)} - 2\}\right] \\ &= \frac{2}{s} + e^{-s}\left(8e^{-(s+1)}\frac{1}{s+1} - \frac{2}{s}\right) \\ &= \boxed{\frac{2}{s}\left(1 - e^{-s}\right) + \frac{8}{s+1}e^{-(s+1)}} \end{aligned}$$

(b) Using that $\mathcal{L}\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}$, we get

$$-\frac{7}{(s-1)^2+4} = -\frac{7}{2}\cdot\frac{2}{(s-1)^2+2^2} = \mathcal{L}\left\{-\frac{7}{2}e^t\sin(2t)\right\}$$

and

$$\frac{8}{(s-1)^2+5} = \frac{8}{\sqrt{5}}\cdot\frac{\sqrt{5}}{(s-1)^2+5} = \mathcal{L}\left\{\frac{8}{\sqrt{5}}e^t\sin(\sqrt{5}t)\right\}.$$

Thus, the inverse Laplace transform of $F(s)$ is

$$\boxed{f(t) = -\frac{7}{2}e^t\sin(2t) + \frac{8}{\sqrt{5}}e^t\sin(\sqrt{5}t)}.$$