APPM 2360: Midterm exam 2

October 24, 2018   1.5 hours

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your instructor's name, (3) your section number and (4) a grading table. Text books, class notes, cell phones and calculators are NOT permitted. A one page (letter sized **1 side only**) crib sheet is allowed.

Problem 1: (20 points) Are the following statements true or false? (Only answer TRUE or FALSE is needed; no justification is required.) If a statement is not always true, reply 'FALSE'. Give your answers in the bluebook (and not on this problem sheet).

(a) The vectors $\vec{\mathbf{v_1}} = (1, 0, 0, 1)$, $\vec{\mathbf{v_2}} = (0, 1, 0, 0)$ and $\vec{\mathbf{v_3}} = (0, 0, 1, 0)$ span $\mathbb{R}^4$.

(b) The dimension of the space of diagonal $n \times n$ real matrices with respect to the usual matrix addition and multiplication by real numbers is n^2.

(c) If $\{v_1, v_2, v_3\}$ is a basis of a vector space V, then v_1 and v_3 are linearly independent.

(d) If A is an $m \times n$ matrix and RREF(A) has some zero rows,
then rank(A) is always less than n.

(e) Every nonzero $m \times n$ matrix has at least one pivot.

Solution:

(a) F

(b) F

(c) T

(d) F

(e) T

Problem 2: (20 points)

(a) Given n functions $f_1(x)$, $f_2(x)$, $\ldots$, $f_n(x)$, give the definition for these functions being linearly independent.

(b) Consider the specific case of the functions $\{x, 1 + x, 2 + 3x\}$. Determine whether these three functions are linearly independent, or not.

(c) Define the Wronskian $W(x)$ of a general set of n functions $f_1(x)$, $f_2(x)$, $\ldots$, $f_n(x)$.

(d) Apply the Wronskian to the set of functions listed in part (b) of this problem. What can you conclude from the resulting function $W(x)$ regarding linear independence?

Solution:

(a) The functions are linearly independent if and only if the relation $c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) \equiv 0$ (identically equal to zero) for constants $c_1, c_2, \ldots, c_n$ can only be satisfied by $c_1 = c_2 = \cdots = c_n = 0$.

(b) The relation to test becomes in this case $c_1 x + c_2(1+x) + c_3(2+3x) \equiv 0$, which we rewrite as $(c_2 + 2c_3) \cdot 1 + (c_1 + c_2 + 3c_3) \cdot x \equiv 0$. For this to hold, the coefficients for both 1 and x must be zero, i.e.

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is a non-square linear system, so it has either no or infinitely many solutions. No solution cannot be the case, since we know the system to have the zero solution. It must therefore also have nontrivial solution(s), confirming linear dependence. (One can easily see that, for example, $c_1 = 1$, $c_2 = 2$, $c_3 = -1$ solves the system, but finding an explicit solution is unnecessary for solving the present problem).

(c)

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & f_n'(x) \\ \ldots & \ldots & \ldots & \ldots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \ldots & f_n^{(n-1)}(x) \end{vmatrix}$$

(d) The Wronskian becomes

$$W(x) = \begin{vmatrix} x & 1+x & 2+3x \\ 1 & 1 & 3 \\ 0 & 0 & 0 \end{vmatrix} \equiv 0.$$

Had $W(x)$ not been identically zero, we could have concluded that the functions are independent. With $W(x) \equiv 0$, this test using the Wronskian is inconclusive.

Problem 3: (20 points) Consider the following matrices:

$$A = \begin{pmatrix} 0 & 0 & 2 \\ -1 & 1 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 3 & 0 \\ -4 & 2 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

Find the following quantities:

(a) $C + 2B$

(b) AB^T

(c) The eigenvalues of B and their multiplicities.

(d) The eigenvalues <u>and</u> eigenvectors of D.

Solution:

(a)

$$C + 2B = \begin{pmatrix} 3 & 2 & 2 \\ -4 & 5 & 2 \\ -4 & 2 & 3 \end{pmatrix}$$

(b)

$$AB^T = \begin{pmatrix} 0 & 0 & 2 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 3 & 2 \end{pmatrix}$$

(c) Solve equation $\det(B - \lambda I) = 0$ for λ i.e.

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

which is $(1-\lambda)^3 = 0$. Thus, $\lambda = 1$ with multiplicity 3 or $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Alternatively, the eigenvalues of an upper (or lower) triangular matrix can be read off from the diagonal elements, giving the same result.

(d) The equation $\det(D - \lambda I) = 0$ is $(3-\lambda)^2 - 1 = 0$, so $\lambda - 3 = \pm 1$ and eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 2$. Solving $(D - \lambda_1 I)\vec{\mathbf{v_1}} = \vec{\mathbf{0}}$ for $\vec{\mathbf{v_1}} = (x, y)^T$, we find

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0,$$

i.e. $x = y$ and $\vec{\mathbf{v_1}} = x(1, 1)^T$ is an eigenvector for any $x \neq 0$.
Similarly, $(D - \lambda_2 I)\vec{\mathbf{v_2}} = \vec{\mathbf{0}}$ with $\vec{\mathbf{v_2}} = (x, y)^T$, is

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0,$$

i.e. $x = -y$ and $\vec{\mathbf{v_2}} = x(1, -1)^T$ is an eigenvector for any $x \neq 0$.

Problem 4: (20 points)

(a) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & 6 \\ 0 & 0 & 1 \end{bmatrix}$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & -2 & 6 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 7 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow -1 \cdot R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -7 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_1 \leftarrow R_1 - R_3]{R_2 \leftarrow R_2 + 7 \cdot R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -8 \\ 0 & 1 & 0 & -1 & -1 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

so the answer is $A^{-1} = \begin{bmatrix} 2 & 1 & -8 \\ -1 & -1 & 7 \\ 0 & 0 & 1 \end{bmatrix}$.

Alternatively, A^{-1} can be found as $(A^{-1})_{ij} = C_{ji}/\det(A)$ in terms of cofactors C_{ij} of each element A_{ij} of A in $\det(A)$.

(b) Solve the following system of equations for x and y:

$$x + y = 2$$
$$5x + 4y = 3$$

Hint: observe $\begin{bmatrix} 1 & 1 \\ 5 & 4 \end{bmatrix} \cdot \begin{bmatrix} -4 & 1 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Solution: Observe that we can write the system of equations as $\begin{bmatrix} 1 & 1 \\ 5 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. The hint tells us that the inverse of the coefficient matrix is $\begin{bmatrix} -4 & 1 \\ 5 & -1 \end{bmatrix}$, so we just compute $\begin{bmatrix} -4 & 1 \\ 5 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$.

Problem 5: (20 points) This problem consists of three independent parts.

(a) Consider the set S of all functions $f(x)$ continuous on $[0,1]$ and such that $f(0) = f(1)$. Is this a vector space? Justify your answer.

(b) Express the given vector $\vec{\mathbf{v}}$ in terms of the given basis $\{\vec{\mathbf{v_1}}, \vec{\mathbf{v_2}}\}$ of the vector space $\mathbb{R}^2$, where

$$\vec{\mathbf{v}} = \begin{pmatrix} a \\ b \end{pmatrix}, \qquad \vec{\mathbf{v_1}} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \vec{\mathbf{v_2}} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

(c) Consider $\mathcal{P}_4$, the vector space of all polynomials $p(x)$ of degree ≤ 4. Are $p_1(x) = x^3 - 3x + 1$, $p_2(x) = x^4 - 6x + 3$ and $p_3(x) = x^4 - 2x^3 + 1$ linearly independent elements of $\mathcal{P}_4$? What is the dimension of the subspace of $\mathcal{P}_4$ they span? Give an example of a basis for this subspace. Justify your answers.

Solution:

(a) This set is a vector space if any linear combination of two such functions is also in S. Let $f_1(x) \in S$ and $f_2(x) \in S$ be two such functions and c_1, c_2 be two numbers. Then function $f(x) = c_1 f_1(x) + c_2 f_2(x)$ is continuous on $[0,1]$ and

$$f(0) = c_1 f_1(0) + c_2 f_2(0) = c_1 f_1(1) + c_2 f_2(1) = f(1),$$

so $f(x) \in S$. Therefore S is a vector space.

(b) We need to find numbers x_1 and x_2 such that $\vec{\mathbf{v}} = x_1 \vec{\mathbf{v_1}} + x_2 \vec{\mathbf{v_2}}$ i.e. to solve the linear system $A\vec{\mathbf{x}} = \vec{\mathbf{v}}$ where

$$\vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad A = (\vec{\mathbf{v_1}} | \vec{\mathbf{v_2}}) = \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix}$$

Matrix A is invertible since $\det A = 8$, therefore

$$\vec{\mathbf{x}} = A^{-1}\vec{\mathbf{v}} = \frac{1}{8}\begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 2a+b \\ -2a+3b \end{pmatrix}$$

Thus,

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{(2a+b)}{8}\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \frac{(-2a+3b)}{8}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

(c) Consider numbers c_1, c_2, c_3 such that $c_1(x^3 - 3x + 1) + c_2(x^4 - 6x + 3) + c_3(x^4 - 2x^3 + 1) = 0$ for all x, then we get the system

$$\begin{cases} c_2 + c_3 = 0 \\ c_1 - 2c_3 = 0 \\ -3c_1 - 6c_2 = 0 \\ c_1 + 3c_2 + c_3 = 0 \end{cases}$$

We verify that, if $c_2 = -c_3$ and $c_1 = 2c_3$, then the last two equations are also satisfied. Thus, one of the three numbers is arbitrary and the polynomials are linearly dependent. However, e.g. p_1 and p_2 are linearly independent (they have different powers of x). Therefore

$$\text{span}\{p_1, p_2, p_3\} = \text{span}\{p_1, p_2\}$$

and the dimension $\dim(\text{span}\{p_1, p_2\}) = 2$. $\{p_1, p_2\}$ is a possible basis for this subspace. Alternatively, we can use the (general method of) Gaussian elimination to solve the linear system $A\vec{\mathbf{c}} = \vec{\mathbf{0}}$ for $\vec{\mathbf{c}} = (c_1, c_2, c_3)^T$. The matrix of the coefficients is

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -2 \\ -3 & -6 & 0 \\ 1 & 3 & 1 \end{pmatrix}$$

and we find RREF(A) with the same result.