Problem 1 (18 pts): Let $m(t)$ be the amount of mass in kg of a radioactive substance, and assume dm/dt is proportional to m due to radioactive decay. The amount of substance is measured at 3 kg at $t = 0$. One minute later the substance is measured at 2 kg.

(a) (10 pts) How much substance remains after another minute has passed (i.e., two minutes into the experiment)?

(b) (8 pts) Use Euler's method with step size $h = 1$ to approximate the remaining amount of radioactive substance at $t = 1$ and $t = 2$ starting from $t = 0$. There is no need to approximate logarithms.

Solution:

(a) First we write down the governing differential equation described,

$$\frac{dm}{dt} = km.$$

Solving by separation of variables,

$$\int \frac{dm}{m} = \int k dt$$

we arrive at $m(t) = Ae^{kt}$. Applying the initial condition, $m(0) = A = 3$kg. To find $m(2)$ we first need to find k. Use the amount of material left after one minute,

$$m(1) = 2 = 3e^k.$$

This gives us that $k = \ln\left(\frac{2}{3}\right)$. Finally, plugging in $t = 2$ we find the amount remaining two minutes into the experiment,

$$m(2) = 3e^{2\ln\left(\frac{2}{3}\right)} = 3e^{\ln\left(\frac{4}{9}\right)} = \frac{12}{9} = \frac{4}{3}$$

(b) From part (a) we found $k = \ln\left(\frac{2}{3}\right)$. Thus, $f(m, t) = \ln\left(\frac{2}{3}\right)$ and $m_0 = 3$. The first iteration of Euler's method with step size $h = 1$ will give us an approximation at $t = 1$,

$$\begin{aligned} m_1 &= m_0 + h\left(f(m_0, t_0)\right) \\ &= 3 + 1\left(3\ln\left(\frac{2}{3}\right)\right) \\ &= 3\left(1 + \ln\left(\frac{2}{3}\right)\right). \end{aligned}$$

The second iteration of Euler's will approximate the solution at $t = 2$,

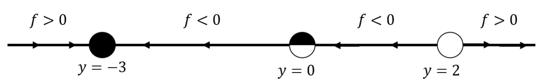
$$\begin{aligned} m_2 &= m_1 + h\left(f(m_1, t_1)\right) \\ &= 3\left[1 + \ln\left(\frac{2}{3}\right)\right] + 1\left(\ln\left(\frac{2}{3}\right)\left[3\left(1 + \ln\left(\frac{2}{3}\right)\right)\right]\right) \\ &= 3\left[1 + \ln\left(\frac{2}{3}\right) + \ln\left(\frac{2}{3}\right) + \left(\ln\left(\frac{2}{3}\right)\right)^2\right] \\ &= 3\left(1 + \ln\left(\frac{2}{3}\right)\right)^2 \end{aligned}$$

Problem 2 (20 pts): Solution:

	$y' = y^{1/3}t$	$y' = -y + t$	$ty' + y = 0$	$ty' + y = 2$	$y' = \sqrt{\lvert y \rvert}$
Matching direction field from Fig. **??**	D	E	A	B	C
Is the equation linear?	No	Yes	Yes	Yes	No
If linear, is it homogeneous?	-	No	Yes	No	-
If $y_1(t)$ and $y_2(t)$ both solve the ODE, does $y_3(t) = y_1(t) + 7y_2(t)$ also solve the ODE?	No	No	Yes	No	No
Is the equation separable?	Yes	No	Yes	Yes	Yes
Can you guarantee a solution exists for all (t_0, y_0) in R?	Yes	Yes	No	No	Yes
Can you guarantee solutions are unique for all (t_0, y_0) in R?	No	Yes	No	No	No

Problem 3 (16 pts): The follow questions are unrelated. Please answer each question with a short response. Justify your answers.
Solution:

(a) This differential equation is autonomous with $f(y) = (y-2)(y+3)y^2$. We have that $y' = f(y) = 0$ when $y = -3, 0, 2$. Plugging in points into $f(y)$ on either side of each equilibrium solutions gives the stability.



When $y = 1$ we find $f(y) < 0$ and the solution tends towards $y = 0$ and away from $y = 2$.

(b) From the differential equation we know the carrying capacity is $L = 1$. Solutions on either side of the line $y = 1$ will behave different.

(i) When the initial condition exists in $(0, 1)$ the solution grows with asymptotic behavior towards $y = 1$.

(ii) When the initial condition exists lies above $y = 1$ the solution decays towards $y = 1$.

Problem 4 (22 pts): Consider an internally uniform spherical shell with inner radius 1m and outer radius 2m. The inner surface is kept at a constant 15°C, while the outer surface is kept at a constant 25°C. The long-term temperature distribution $T(r)$ within the shell $(1 \leq r \leq 2)$ satisfies

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} = 0.$$

Solve this to get an expression for the temperature $T(r)$. Your answer should not involve any unknown constants. Hint: Let $S = \frac{dT}{dr}$.

Solution: First we let $S = \frac{dT}{dr}$. Notice, $\frac{d^2T}{dr^2} = \frac{d}{dr}\left(\frac{dT}{dr}\right) = \frac{d}{dr}S = \frac{dS}{dr}$. The differential equation is then,

$$\frac{dS}{dr} + \frac{2}{r}S = 0.$$

This can be solved with separation of variables,

$$\int \frac{dS}{S} = \int \frac{-2}{r} dr \qquad \Longrightarrow \qquad \ln|S| = -2\ln|r| + C_1 \qquad \Longrightarrow \qquad S = Ar^{-2}$$

This gives us that $\frac{dT}{dt} = Ar^{-2}$. Separation of variables again,

$$\int dT = \int Ar^{-2} dr \qquad \Longrightarrow \qquad T = -Ar^{-1} + C_2$$

There are now two constants to solve for with our two conditions, $T(1) = 15$ and $T(2) = 25$. This gives two equations,

$$-A + C_2 = 15$$
$$-\frac{A}{2} + C_2 = 25$$

Which gives $A = 20$ and $C_2 = 35$. Finally,

$$\boxed{T(r) = -\frac{20}{r} + 35}$$

Problem 5 (24 pts) The governing equations for $A(t)$ and $B(t)$ are

$$\frac{dA}{dt} = -\frac{A}{10}$$
$$\frac{dB}{dt} = \frac{1}{10}(A - B).$$

(a) (12 pts) Solve for $A(t)$ and $B(t)$.

(b) (6 pts) Describe in general terms what the solutions, $A(t)$ and $B(t)$, look like (for example what are their initial slopes and long term behavior?).

(c) (6 pts) A pump is added to T_B which pumps 1 L/ min into T_A. To keep the water levels the same, the flow rate from T_A into T_B is increased to 2 L/ min. Write down the governing differential equations (you do not need to solve). Sketch the nullclines (and directional arrows), and label any equilibrium points and the initial condition.

Solution:

(a) To find the solution we first need to solve for $A(t)$. We can do so by separation of variables,

$$\int \frac{dA}{A} = \int -\frac{dt}{10} \qquad \Longrightarrow \qquad A(t) = Ce^{-\frac{t}{10}}$$

Using the initial condition, $A(0) = 2$, we find $A(t) = 2e^{-\frac{t}{10}}$. the differential equation for $B(t)$ then is,

$$\frac{dB}{dt} + \frac{1}{10}B = \frac{1}{5}e^{-\frac{t}{10}}$$

This can be solved with either integrating factor or Euler-Lagrange,
Integrating Factor:

$$\mu(t) = e^{\int \frac{1}{10}dt} = e^{\frac{t}{10}} \qquad \Longrightarrow \qquad e^{\frac{t}{10}}\left[\frac{dB}{dt} + \frac{1}{10}B\right] = \frac{d}{dt}\left[e^{\frac{t}{10}}B\right] = e^{\frac{t}{10}}\left(\frac{1}{5}e^{-\frac{t}{10}}\right)$$

$$\Longrightarrow \qquad e^{\frac{t}{10}}B = \int \frac{1}{5}dt = \frac{t}{5} + C$$

Using the initial condition, $B(0) = 5$, we find $C = 5$ and

$$B(t) = e^{-\frac{t}{10}}\left(\frac{t}{5} + 5\right)$$

<u>Euler-Lagrange:</u>
The homogeneous solution solves,

$$\frac{dB}{dt} = -\frac{B}{10}$$

which gives $B_h = Ce^{-\frac{t}{10}}$. Now assuming $B_p = v(t)e^{-\frac{t}{10}}$ we can find the relation,

$$v(t)'e^{-\int \frac{1}{10}dt} = f(t) \implies v'e^{-\frac{t}{10}} = \frac{1}{5}e^{-\frac{t}{10}} \implies v(t) = \frac{t}{5}$$

The solution is then,

$$B(t) = B_h + B_p = Ce^{-\frac{t}{10}} + \frac{t}{5}e^{-\frac{t}{10}}$$

Using the initial condition, $B(0) = 5$ we find $C = 5$ and

$$B(t) = e^{-\frac{t}{10}}\left(\frac{t}{5} + 5\right)$$

(b) The initial slopes are $\left.\frac{dA}{dt}\right|_{t=0} = -\frac{1}{5}$ and $\left.\frac{dB}{dt}\right|_{t=0} = -\frac{3}{10}$. The amount of salt in both tanks are always decreasing exponentially until all the salt has left the tanks.

(c) The governing equations become,

$$\frac{dA}{dt} = \frac{1}{10}(B - 2A)$$
$$\frac{dB}{dt} = \frac{1}{5}(A - B)$$

To sketch the phase plane we first note $\frac{dA}{dt} = 0$ when $B = 2A$ and $\frac{dB}{dt} = 0$ when $B = A$.