1. [2350/072624 (24 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

(a) The function $f(x, y) = 5x + y^{1/3}$ has no critical points.

(b) $x^2 - 2y^2 + 3z^2 + 4x = -1$ is a hyperboloid of two sheets.

(c) The planes $x + y - 3z = 4$ and $-5x - 5y = -15z$ do not intersect.

(d) If you are standing on the surface $f(x, y) = 1 - x^2 - y^3$ at the point $(1, -1, 1)$ and begin walking in the direction $6\,\mathbf{i} - 4\,\mathbf{j}$, you will be following a level curve of $f(x, y)$.

(e) The acceleration vector of the curve $\mathbf{r}(t) = -2\sin t\,\mathbf{j} - 2\cos t\,\mathbf{k}$ lies in the xz-plane.

(f) The symmetric equations of the line through the origin in the direction of $\langle 1, 1, 1\rangle$ are $x = y = z$.

(g) The vector field $\mathbf{V}(x, y, z) = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$ is both irrotational and incompressible.

(h) For all $(x_0, y_0) \in \mathbb{R}^2$, $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{\cos x + \sin y}{1 + x^2 + y^2} = \frac{\cos x_0 + \sin y_0}{1 + x_0^2 + y_0^2}$.

SOLUTION:

(a) **FALSE** $f_x = 5$ and $f_y = \dfrac{1}{3}y^{-2/3}$. Since $f_y(0, 0)$ does not exist, $(0, 0)$ is a critical point.

(b) **FALSE** $x^2 + 4x + 4 - 4 - 2y^2 + 3z^2 = -1 \implies (x + 2)^2 - 2y^2 + 3z^2 = 3$ is a hyperboloid of one sheet.

(c) **TRUE** The planes' normal vectors are $\langle 1, 1, -3\rangle$ and $\langle -5, -5, 15\rangle$ which are scalar multiples of one another, implying that they and the planes are parallel and thus do not intersect.

(d) **TRUE** $\nabla f(x, y) = \langle -2x, -3y^2\rangle \implies \nabla f(1, -1) = \langle -2, -3\rangle$ and since $\langle -2, -3\rangle \cdot \langle 6, -4\rangle = 0$, the directional derivative is zero implying that you are following a level curve.

(e) **FALSE** Since the curve lies in the yz-plane, its derivatives, namely $a = \mathbf{r}''(t)$, do as well.

(f) **TRUE** The line's parametric equations are $x = t, y = t, z = t$.

(g) **TRUE**

$$\nabla \cdot \mathbf{V} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0$$

$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz & xy \end{vmatrix} = (x - x)\,\mathbf{i} + (y - y)\,\mathbf{j} + (z - z)\,\mathbf{k} = \mathbf{0}$$

(h) **TRUE** Since the numerator and denominator are both continuous functions throughout $\mathbb{R}^2$ and the denominator is never zero, the function is continuous throughout $\mathbb{R}^2$.

■

2. [2350/072624 (24 pts)] Let $f(x, y) = \tan^{-1}\dfrac{y}{x}$ and consider the path, $\mathcal{C}$, given by $\mathbf{r}(t) = \sqrt{2}\sin t\,\mathbf{i} + \sqrt{2}\cos t\,\mathbf{j}$, $\dfrac{\pi}{4} \le t \le \dfrac{3\pi}{4}$. Recall that $\left(\tan^{-1} x\right)' = (1 + x^2)^{-1}$.

(a) (10 pts) Show that $f(x, y)$ is a potential function for $\mathbf{E} = -\dfrac{y}{x^2 + y^2}\,\mathbf{i} + \dfrac{x}{x^2 + y^2}\,\mathbf{j}$.

(b) (10 pts) Find the work done moving an object through the vector field along the given path.

(c) (4 pts) Find the work done moving an object around any ellipse located completely in the fourth quadrant.

SOLUTION:

(a) We need to show that $\mathbf{E} = \nabla f$.

$$\frac{\partial f}{\partial x} = \frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

$$\frac{\partial f}{\partial y} = \left[\frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right)\right]\left(\frac{x}{x}\right) = \frac{x}{x^2+y^2}$$

$$\implies \mathbf{E} = \nabla f$$

(b) Use the Fundamental Theorem for Line Integrals. The oriented path runs from $(1,1)$ to $(1,-1)$. Thus

$$\text{Work} = \int_{\mathcal{C}} \mathbf{E}\cdot \mathrm{d}\mathbf{r} = \int_{(1,1)}^{(1,-1)} \nabla f \cdot \mathrm{d}\mathbf{r} = f(1,-1) - f(1,1) = \tan^{-1}\left(\frac{-1}{1}\right) - \tan^{-1}\left(\frac{1}{1}\right) = -\frac{\pi}{4} - \frac{\pi}{4} = -\frac{\pi}{2}$$

(c) Since any ellipse is a closed curve and we are dealing with a conservative vector field in a simply-connected domain (Quad IV), the work done is 0.

■

3. [2350/072624 (48 pts)] Consider a metal plate consisting of the first octant portion of the cylinder $x - y = 0$, $0 \le z \le 2$, $0 \le x \le 1$. The plate is made of a metal alloy whose density is $\delta(x,y,z) = x + y + 1$.

(a) (16 pts) Find the mass of the plate.

(b) (16 pts) Using a normal vector to the plate oriented with a positive **i**- and negative **j**-component, show that the flux of $\mathbf{F} = (x+y)\,\mathbf{i} - \mathbf{j} + z\,\mathbf{k}$ through the plate equals the mass of the plate divided by $\sqrt{2}$. Hint: take advantage of the work you did in part (a).

(c) (16 pts) Suppose the plate represents a wall on a building. The height of the roof of the building is $z = 1 + x + y$ and the wall extends from the xy-plane up to the roof. Use an appropriate integral to find the area of one side of the wall.

SOLUTION:

(a) We need to compute a scalar surface integral.

$$g(x,y,z) = x - y \implies \nabla g = \langle 1, -1, 0\rangle \implies \|\nabla g\| = \sqrt{2}$$

$$\text{project onto } xz\text{-plane} \implies \mathbf{p} = \mathbf{j},\ |\nabla g \cdot \mathbf{p}| = |-1| = 1 \text{ and } \mathcal{R} \text{ is } 0 \le x \le 1, 0 \le z \le 2$$

$$\text{Mass} = \iint_{\mathcal{S}} \delta(x,y,z)\,\mathrm{d}S = \int_0^1\int_0^2 (1+x+y)\sqrt{2}\,\mathrm{d}z\,\mathrm{d}x \quad (\text{eliminate } y \text{ using the surface})$$

$$= \sqrt{2}\int_0^1\int_0^2 (1+2x)\,\mathrm{d}z\,\mathrm{d}x = \sqrt{2}\int_0^1 2(1+2x)\,\mathrm{d}x = 2\sqrt{2}\left(x+x^2\right)\Big|_0^1 = 4\sqrt{2}$$

(b) The given information requires using $+\nabla g$ for **n** and we have $\mathbf{F}\cdot\mathbf{n} = \langle x+y, -1, z\rangle \cdot \langle 1, -1, 0\rangle = 1 + x + y$

$$\text{Flux} = \iint_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \int_0^1\int_0^2 (1+x+y)\,\mathrm{d}z\,\mathrm{d}x = 4 = \frac{\text{Mass}}{\sqrt{2}}$$

(c) The bottom of the wall is the curve $\mathcal{C}$ given by $y = x$ which is parameterized as $\mathbf{r}(t) = \langle t, t\rangle$, $0 \le t \le 1$ with $\mathbf{r}'(t) = \langle 1, 1\rangle$ and $\|\mathbf{r}'(t)\| = \sqrt{2}$.

$$\text{Area} = \int_{\mathcal{C}} (1+x+y)\,\mathrm{d}s = \int_0^1 (1+2t)\sqrt{2}\,\mathrm{d}t = \sqrt{2}\left(t+t^2\right)\Big|_0^1 = 2\sqrt{2}$$

■

4. [2350/072624 (16 pts)] Use Stokes' theorem to evaluate $\iint_{\mathcal{S}} \nabla\times\mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S$ where $\mathcal{S}$ is given by $x^2+y^2-z^2 = -1$, $-\sqrt{5} \le z \le -1$, upward pointing normal, and $\mathbf{V} = (y-x)\,\mathbf{i} - (x+y)\,\mathbf{j} + e^{xyz}\,\mathbf{k}$.

SOLUTION:

The boundary, $\partial\mathcal{S}$, of the surface, $\mathcal{S}$, is the circle $x^2+y^2=4$, oriented counterclockwise when looking down, given the orientation of the surface.

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} - \sqrt{5}\,\mathbf{k},\ 0 \le t \le 2\pi$$

$$\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + 0\,\mathbf{k}$$

$$\mathbf{V}[\mathbf{r}(t)] = (2\sin t - 2\cos t)\,\mathbf{i} - (2\cos t + 2\sin t)\,\mathbf{j} + e^{-4\sqrt{5}\cos t\sin t}\,\mathbf{k}$$

$$\mathbf{V}[\mathbf{r}(t)]\cdot\mathbf{r}'(t) = -4\sin^2 t + 4\cos t\sin t - 4\cos^2 t - 4\cos t\sin t = -4$$

$$\iint_{\mathcal{S}} \nabla\times\mathbf{V}\cdot\mathbf{n}\,\mathrm{d}S = \int_{\partial S}\mathbf{V}\cdot\mathrm{d}\mathbf{r} = \int_0^{2\pi} -4\,\mathrm{d}t = -8\pi$$

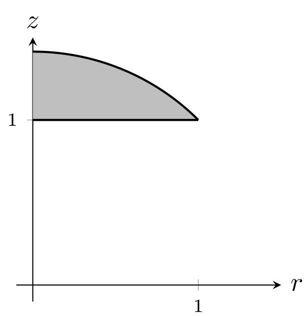
5. [2350/072624 (22 pts)] Consider the solid region, $\mathcal{W}$, enclosed below the sphere $x^2+y^2+z^2=2$ and above the plane $z=1$ and let $\partial\mathcal{W}$ be its boundary.

   (a) (16 pts) Use Gauss' Divergence theorem to find the flux of $\mathbf{F} = e^{yz}\,\mathbf{i} + xz\,\mathbf{j} + 2(z^2-1)\,\mathbf{k}$ through $\partial\mathcal{W}$.

   (b) (6 pts) Briefly explain in words why the flux in part (a) is the same as the flux through just the spherical portion of $\partial\mathcal{W}$.

SOLUTION:

(a) A sketch of a portion of $\mathcal{W}$ in a constant θ plane is shown below.



$$\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}\left(e^{yz}\right) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}\left[2\left(z^2-1\right)\right] = 4z$$

$$\text{Spherical:}\quad \text{Flux} = \iint_{\partial\mathcal{W}} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \iiint_{\mathcal{W}} \nabla\cdot\mathbf{F}\,\mathrm{d}V = \int_0^{2\pi}\int_0^{\pi/4}\int_{\sec\phi}^{\sqrt{2}} 4\rho^3\cos\phi\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

$$= 2\pi\int_0^{\pi/4}\left(4\cos\phi\sin\phi - \sec^3\phi\sin\phi\right)\mathrm{d}\phi \overset{u=\cos\phi}{=} 2\pi\int_1^{\sqrt{2}/2}\left(-4u+u^{-3}\right)\mathrm{d}u$$

$$= 2\pi\left(-2u^2 - \frac{1}{2}u^{-2}\right)\Big|_1^{\sqrt{2}/2} = 2\pi\left[(-1-1) - \left(-2-\frac{1}{2}\right)\right] = \pi$$

$$\text{Cylindrical:}\quad \text{Flux} = \iint_{\partial\mathcal{W}} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \iiint_{\mathcal{W}} \nabla\cdot\mathbf{F}\,\mathrm{d}V = \int_0^{2\pi}\int_0^{1}\int_{1}^{\sqrt{2-r^2}} 4zr\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= 2\pi\int_0^1 2\left(r-r^3\right)\mathrm{d}r = 4\pi\left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_0^1 = 4\pi\left(\frac{1}{2}-\frac{1}{4}\right) = \pi$$

(b) Since the $\mathbf{k}$-component of the vector field is 0 when $z=1$ there is no component of $\mathbf{F}$ normal to the plane and thus no flux through the plane. Since the flux through the boundary of ∂W equals the sum of the flux through the sphere plus that through the plane, the flux through $\partial\mathcal{W}$ equals the flux through the sphere.

6. [2350/072624 (16 pts)] Use Green's theorem to find the circulation of $\mathbf{F} = 3xy\,\mathbf{i} + y^2\,\mathbf{j}$ around the semicircle $x^2+y^2=9,\ x\le 0$, oriented counterclockwise.

SOLUTION:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}\left(y^2\right) - \frac{\partial}{\partial y}\left(3xy\right) = -3x$$

$$\begin{aligned}
\text{Circulation} &= \int_{\partial D} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A = \int_{-3}^{0} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} -3x\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{\pi/2}^{3\pi/2} \int_0^3 -3r^2 \cos\theta\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_{\pi/2}^{3\pi/2} -r^3\Big|_0^3 \cos\theta\,\mathrm{d}\theta \\
&= -27\sin\theta\Big|_{\pi/2}^{3\pi/2} = 54
\end{aligned}$$

■