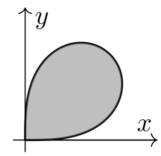
1. [2350/071224 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) The integrals $\int_0^1 \int_0^{\sqrt{1-x^2}} \left(1 - x^2 - y^2\right) \mathrm{d}y\,\mathrm{d}x$ and $\int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^{1-x^2-y^2} \mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$ compute the same quantity.

   (b) For all $f(x,y)$ that are continuous on $\mathbb{R}^2$, $\int_0^1 \int_{-y}^{y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{-1}^{0} \int_0^{-x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x + \int_0^1 \int_0^x f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$

   (c) The area of the following figure is given by $\int_0^{2\pi} \int_0^{\sqrt{\sin 2\theta}} \mathrm{d}r\,\mathrm{d}\theta.$



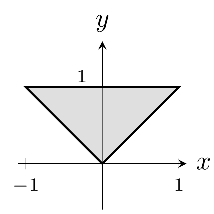
   (d) $(x,y,z) = (-2, 2, -2\sqrt{2})$ and $(\rho, \theta, \phi) = \left(4, \frac{7\pi}{4}, \frac{3\pi}{4}\right)$ represent the same point in $\mathbb{R}^3$.

   (e) $\rho = -2\cos\phi$ is a sphere of radius 1 centered at $(0, 0, -1)$.

SOLUTION:

(a) **TRUE** They both compute the volume of the region above the first quadrant portion of the unit circle below the surface $z = 1 - x^2 - y^2$.

(b) **FALSE** Here is a sketch of the region of integration



$$\int_0^1 \int_{-y}^{y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{-1}^{0} \int_{-x}^{1} f(x,y)\,\mathrm{d}y\,\mathrm{d}x + \int_0^1 \int_x^1 f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

(c) **FALSE** Area $= \int_0^{\pi/2} \int_0^{\sqrt{\sin 2\theta}} r\,\mathrm{d}r\,\mathrm{d}\theta$

(d) **FALSE**

$$\rho = \sqrt{(-2)^2 + 2^2 + \left(2\sqrt{2}\right)^2} = 4$$

$$\tan^1\left(\frac{2}{-2}\right) = -\frac{\pi}{4} \quad \text{but need to be in correct quadrant so add } \pi \text{ to get } \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$\phi = \cos^{-1}\left(-\frac{2\sqrt{2}}{4}\right) = \frac{3\pi}{4}$$

OR

$$x = 4\sin\frac{3\pi}{4}\cos\frac{7\pi}{4} = 4\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = 2$$

$$y = 4\sin\frac{3\pi}{4}\sin\frac{7\pi}{4} = 4\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = -2$$

$$z = 4\cos\left(\frac{3\pi}{4}\right) = -2\sqrt{2}$$

(e) **TRUE**

$$\sqrt{x^2+y^2+z^2} = -2\frac{z}{\sqrt{x^2+y^2+z^2}}$$
$$x^2+y^2+z^2+2z+1-1=0$$
$$x^2+y^2+(z+1)^2=1$$

■

2. [2350/071224 (20 pts)] After the first bite, a slice of pizza occupies the region, $\mathcal{D}$, in Quadrant II, bounded by the lines $y=-x$ and $x=0$ and the arc of the circles $x^2+y^2=9$ and $x^2+y^2=1$. The density of the pizza is $\delta(x,y)=\dfrac{6x^2}{x^2+y^2}$. If the mass of the pizza slice is $3(\pi-2)$, find $\bar{y}$, the y-coordinate of its center of mass.

SOLUTION:

This is an obvious candidate for polar coordinates. With that said, $\delta(r,\theta)=\dfrac{6r^2\cos^2\theta}{r^2}=6\cos^2\theta$. The pizza occupies the polar region given by $1\le r\le 3, \pi/2\le\theta\le 3\pi/4$. We need to compute the moment around the x-axis, $M_x=\iint_{\mathcal{D}} y\delta(x,y)\,\mathrm{d}A$. This is

$$M_x=\int_{\pi/2}^{3\pi/4}\int_1^3 (r\sin\theta)\left(6\cos^2\theta\right) r\,\mathrm{d}r\,\mathrm{d}\theta = \left(\int_1^3 r^2\,\mathrm{d}r\right)\left(\int_{\pi/2}^{3\pi/4} 6\cos^2\theta\sin\theta\,\mathrm{d}\theta\right) \qquad u=\cos\theta$$

$$=\left(\frac{r^3}{3}\bigg|_1^3\right)\left(\int_{-\sqrt{2}/2}^{0} 6u^2\,\mathrm{d}u\right)=\frac{26}{3}\left(\frac{6u^3}{3}\bigg|_{-\sqrt{2}/2}^{0}\right)=\frac{26}{3}\left[2\left(\frac{2\sqrt{2}}{8}\right)\right]=\frac{13\sqrt{2}}{3}$$

$$\Longrightarrow \bar{y}=\frac{M_x}{\text{mass}}=\frac{13\sqrt{2}/3}{3(\pi-2)}=\frac{13\sqrt{2}}{9(\pi-2)}$$

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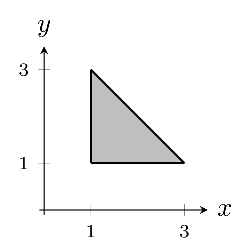
3. [2350/071224 (23 pts)] Use the change of variables $u=x-y,\ v=x+y$ to evaluate $\displaystyle\int_1^3\int_1^{4-x}\left(\frac{x-y}{x+y}\right)\mathrm{d}y\,\mathrm{d}x$.

SOLUTION:

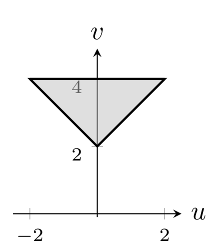
The region of integration is



To find the new integration region, since the transformation is linear, $T(x,y)=(x-y,x+y)$, we find the transformation of the vertices and then connect these with lines.

$$T(1,1)=(0,2);\quad T(3,1)=(2,4);\quad T(1,3)=(-2,4)$$

so the new region is



Next, $x = \dfrac{u+v}{2}$, $y = \dfrac{v-u}{2}$ and

$$J(u,v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

Then

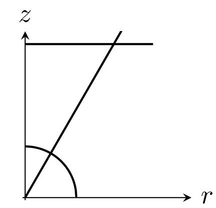
$$\int_1^3 \int_1^{4-x} \left(\frac{x-y}{x+y}\right) \mathrm{d}y\,\mathrm{d}x = \int_2^4 \int_{2-v}^{2+v} \frac{u}{v} \left|\frac{1}{2}\right| \mathrm{d}u\,\mathrm{d}v = \frac{1}{2}\int_2^4 \frac{(2+v)^2 - (2-v)^2}{2v}\,\mathrm{d}v$$

$$= \frac{1}{4}\int_2^4 \frac{4+4v+v^2 - \left(4-4v+v^2\right)}{v}\,\mathrm{d}v = 2\int_2^4 \mathrm{d}v = 4$$

4. [2350/071224 (29 pts)] Let $\mathcal{E}$ be the first octant portion of the solid region that consists of all points satisfying

$$x^2+y^2+z^2 \geq 4 \quad \text{AND} \quad z \leq 6 \quad \text{AND} \quad z \geq \sqrt{3x^2+3y^2}$$

A helpful figure of an arbitrary θ plane is shown below. It is drawn to scale, but you will need to find the specific intersection locations and correctly identify the region of integration based on the three inequalities shown above.



The temperature in the solid is given by $T(x,y,z) = z\left(x^2+y^2\right)$. Fully set up, but **DO NOT EVALUATE**, integral(s) to find:

(a) (15 pts) The *average temperature* of the region using spherical coordinates and integration order $\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$.

(b) (14 pts) The *volume* of the region using cylindrical coordinates and integration order $\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta$

SOLUTION:

A sketch of the actual integration region (shaded) is shown below. The intersection points are found as follows:

$$\text{cone and plane: } 6 = \sqrt{3x^2+3y^2} \implies 6 = \sqrt{3}r \implies r = \frac{6}{\sqrt{3}} = 2\sqrt{3}$$

$$\text{cone and sphere: } x^2+y^2+3x^2+3y^2 = 4 \implies 4r^2 = 4 \implies r = \pm 1 \implies r = 1, z = \sqrt{3}$$

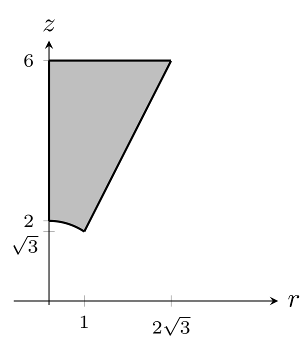
We need to convert the equations of the region into spherical and cylindrical coordinates:

$$\text{sphere: } \rho^2 = 4 \implies \rho = 2; \quad r^2+z^2 = 4 \implies z = +\sqrt{4-r^2} \text{ or } r = \sqrt{4-z^2}$$

$$\text{cone: } \rho\cos\phi = \sqrt{3}\rho\sin\phi \implies \tan\phi = 1/\sqrt{3} \implies \phi = \pi/6; \quad z = \sqrt{3r^2} = \sqrt{3}r \text{ or } r = \frac{z}{\sqrt{3}}$$

$$\text{plane: } \rho\cos\phi = 6 \implies \rho = 6\sec\phi; \quad z = 6$$

The temperature in spherical coordinates is $T(\rho,\theta,\phi) = \rho\cos\phi(\rho\sin\phi)^2 = \rho^3\sin^2\phi\cos\phi$.



(a)

$$T_{avg} = \frac{\iiint_{\mathcal{E}} T(x,y,z)\,\mathrm{d}V}{\iiint_{\mathcal{E}} \mathrm{d}V} = \frac{\int_0^{\pi/2}\int_0^{\pi/6}\int_2^{6\sec\phi} \rho^5 \sin^3\phi\cos\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta}{\int_0^{\pi/2}\int_0^{\pi/6}\int_2^{6\sec\phi} \rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta}$$

(b)

$$\text{Volume} = \int_0^{\pi/2}\int_{\sqrt{3}}^{2}\int_{\sqrt{4-z^2}}^{z/\sqrt{3}} r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta + \int_0^{\pi/2}\int_2^{6}\int_0^{z/\sqrt{3}} r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta$$

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5. [2350/071224 (18 pts)] Evaluate $\displaystyle\int_0^1\int_{\sqrt[3]{z}}^1\int_0^{\ln 3} \frac{\pi e^{2x}\sin(\pi y^2)}{y^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. Hint: $\dfrac{\sin y^2}{y^2}$ has no elementary antiderivative.

SOLUTION:

$$\int_0^1\int_{\sqrt[3]{z}}^1\int_0^{\ln 3} \frac{\pi e^{2x}\sin\left(\pi y^2\right)}{y^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^1\int_{\sqrt[3]{z}}^1 \frac{\pi\sin(\pi y^2)}{y^2}\,\frac{1}{2}e^{2x}\Big|_0^{\ln 3}\,\mathrm{d}y\,\mathrm{d}z = 4\pi\int_0^1\int_{\sqrt[3]{z}}^1 \frac{\sin\left(\pi y^2\right)}{y^2}\mathrm{d}y\,\mathrm{d}z$$

$$\overset{\text{switch}}{\underset{\text{order}}{=}} 4\pi\int_0^1\int_0^{y^3} \frac{\sin\left(\pi y^2\right)}{y^2}\mathrm{d}z\,\mathrm{d}y = 4\pi\int_0^1 y\sin\left(\pi y^2\right)\,\mathrm{d}y \overset{u=\pi y^2}{=} 4\pi\int_0^{\pi}\frac{1}{2\pi}\sin u\,\mathrm{d}u = 2\cos u\Big|_{\pi}^{0} = 4$$

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