1. [2350/062824 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) $f(x,y) = \begin{cases} \dfrac{xy}{x^2+xy+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ is continuous on $\mathbb{R}^2$.

   (b) The level curves of the function $z = 8x^2 - 5y^2$ for all real values of k are parabolas.

   (c) Given $f(u(x,y,z), v(x,y,z))$, the rate of change of f with respect to x is $f_x x_u + f_y y_u$.

   (d) The tangent plane to the surface $e^x + e^y + e^z = 3$ at $(0,0,0)$ is $x + y + z = 0$

   (e) Suppose you move in a direction of the unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$ that makes an angle of $\pi/3$ with the nonzero gradient vector of $f(x,y)$ at a point (x_0, y_0). Then the rate of change of f with respect to distance will be half of the maximum rate of increase of f at that point.

   SOLUTION:

   (a) **FALSE** Away from the origin, the function is a rational function and continuous there. At the origin, the function is defined, but the limit at $(0,0)$ does not exist. Therefore, the function is not continuous there.

   $$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+xy+y^2} = \lim_{(x,mx)\to(0,0)} \frac{mx^2}{x^2+x(mx)+m^2x^2} = \lim_{(x,mx)\to(0,0)} \frac{m}{1+m+m^2} = \frac{m}{1+m+m^2}$$

   which depends on m.

   (b) **FALSE** The level curves are given by $8x^2 - 5y^2 = k$ or $\left(\dfrac{x}{k/8}\right)^2 - \left(\dfrac{y}{k/5}\right)^2 = 1$ for $k \neq 0$ which are hyperbolas. If $k = 0$, the level curves are the lines $y = \pm\sqrt{\frac{8}{5}}x$.

   (c) **FALSE** $f_x = f_u u_x + f_v v_x$

   (d) **TRUE** With $F(x,y,z) = e^x + e^y + e^z$, the normal to the tangent plane is $\nabla F = \langle e^x, e^y, e^z \rangle \implies \nabla F(0,0,0) = \langle 1,1,1 \rangle$ so that the tangent plane is $1(x-0) + 1(y-1) + 1(z-0) = 0$ or $x + y + z = 0$.

   (e) **TRUE** $D_{\mathbf{u}}(x_0,y_0) = \nabla f(x_0,y_0) \cdot \mathbf{u} = \|\nabla f(x_0,y_0)\| \cos(\pi/3) = \frac{1}{2}\|\nabla f(x_0,y_0)\|$ since $\|\nabla f(x_0,y_0)\|$ gives the maximum rate of increase of f at the point.

   ■

2. [2350/062824 (23 pts)] Consider the function $f(x,y) = x^3 + y^3 + x^2y^2$

   (a) (10 pts) Calculate the first order Taylor approximation to $f(x,y)$ centered at the point $(1,1)$.

   (b) (4 pts) Use your result from part (a) to estimate the value of $f(0.9, 1.1)$.

   (c) (9 pts) Now suppose you actually worked out the second order Taylor approximation to $f(x,y)$ centered at $(1,1)$ (you do not actually need to do it). Calculate an upper bound on the absolute value of the error associated with the second order approximation assuming that you only use values of x and y such that $|x-1| \leq 0.1$ and $|y-1| \leq 0.2$.

   SOLUTION:

   (a) $T_1(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

   $$f(1,1) = 3$$
   $$f_x(x,y) = 3x^2 + 2xy^2 \implies f_x(1,1) = 3\left(1^2\right) + 2(1)\left(1^2\right) = 5$$
   $$f_y(x,y) = 3y^2 + 2x^2y \implies f_y(1,1) = 3\left(1^2\right) + 2\left(1^2\right)(1) = 5$$
   $$T_1(x,y) = 3 + 5(x-1) + 5(y-1) = 5x + 5y - 7$$

   (b) $f(0.9, 1.1) \approx T_1(0.9, 1.1) = 3 + 5(0.9-1) + 5(1.1-1) = 3 + 5(-0.1) + 5(0.1) = 3 - 0.5 + 0.5 = 3$

(c) $|E_2(x,y)| \le \dfrac{M}{3!}\left(|x-1|+|y-1|\right)^3$ where M is a bound on the absolute values of the third derivatives of $f(x,y)$ on the region, $\mathcal{R}$, given by $|x-1| \le 0.1, |y-1| \le 0.2$ or $0.9 \le x \le 1.1,\ 0.8 \le y \le 1.2$.

$$f_{xx}(x,y) = 6x + 2y^2 \qquad f_{xxx}(x,y) = 6 \qquad f_{xxy}(x.y) = 4y$$

$$f_{yy}(x,y) = 6y + 2x^2 \qquad f_{yyy}(x,y) = 6 \qquad f_{yyx}(x,y) = 4x$$

$$M = \max_{(x,y)\in\mathcal{R}} \{|f_{xxx}|, |f_{yyy}|, |f_{xxy}|, |f_{yyx}|\} = \max_{(x,y)\in\mathcal{R}} \{6, |4x|, |4y|\}$$

$$= \max\{6, |4(1.1)|, |4(1.2)|\} = \max\{6, 4.4, 4.8\} = 6$$

$$|E_2(x,y)| \le \frac{6}{6}\left(0.1+0.2\right)^3 = \left(\frac{3}{10}\right)^3 = \frac{27}{1000} = 0.027$$

■

3. [2350/062824 (16 pts)] Andrea the ant is crawling around the ground along the path given by $x^2 + y^2 = 2$. The temperature of the ground where she is walking is $T(x,y) = (x-1)^2 + (y-1)^2 + 70$. Use Lagrange multipliers to answer the following questions.

   (a) If Andrea wants to stay away from the warmest spot(s) on the ground, which point(s) should she avoid? What is the temperature of the warmest spot(s)?

   (b) If she wants to chill in the coldest spot(s), where should she go and what will the temperature be there?

SOLUTION:

The objective function to be optimized is $T(x,y)$ and the constraint is $g(x,y) = x^2 + y^2$.

$$T_x = 2(x-1) \qquad T_y = 2(y-1)$$

$$g_x = 2x \qquad g_y = 2y$$

leading to the Lagrange equations

$$2(x-1) = \lambda(2x) \tag{1}$$

$$2(y-1) = \lambda(2y) \tag{2}$$

$$x^2 + y^2 = 2 \tag{3}$$

Since $x \ne 0$ in (1) and $y \ne 0$ in (2), dividing (1) by $2x$ and (2) by $2y$ and equating the results (both equal to λ), we have

$$\frac{x-1}{x} = \frac{y-1}{y} \implies y(x-1) = x(y-1) \implies y = x \qquad \text{[use this in (3)]}$$

$$x^2 + x^2 = 2 \implies x = \pm 1 \implies y = \pm 1$$

The critical points are $(1,1), (-1,-1)$. $T(1,1) = (1-1)^2 + (1-1)^2 + 70 = 70$ and $T(-1,-1) = (-1-1)^2 + (-1-1)^2 + 70 = 78$.

(a) Andrea should avoid the point $(-1,-1)$ where the temperature is 78.

(b) Andrea should chill at the point $(1,1)$ where the temperature is 70.

■

4. [2350/062824 (25 pts)] A charged metal plate occupies the square region of the xy-plane given by $|x| \le 3$, $|y| \le 3$. The charge on the plate is $q(x,y) = 6y^2 - 2y^3 + 3x^2 + 6xy$.

   (a) (5 pts) Find the critical points of the charge function.

   (b) (5 pts) Are there any points, P, on the plate where the charge increases in some directions from P and decreases in others? If so, find them and the value of the charge there. If not, explain why not.

   (c) (5 pts) Are there any points, Q, on the plate where the charge decreases in all directions from Q? If so, find them and the value of the charge there. If not, explain why not.

   (d) (5 pts) Are there any points, R, on the plate where the charge increases in all directions from R? If so, find them and the value of the charge there. If not, explain why not.

(e) (5 pts) Will the plate possess an absolute maximum and minimum charge? Justify your answer. (If the extrema exist, you do not need to find them.)

SOLUTION:

(a)

$$q_x(x,y) = 6x + 6y = 0 \implies y = -x$$

$$q_y(x,y) = 12y - 6y^2 + 6x = 0 \implies 12(-x) - 6(-x)^2 + 6x = 0 \implies x^2 + x = x(x+1) = 0 \implies x = 0, -1$$

$$\text{critical points are } (0,0),\ (-1,1)$$

To answer parts (b), (c), and (d), we need to classify the critical points.

$$q_{xx} = 6 \qquad q_{yy} = 12 - 12y \qquad q_{xy} = 6$$

$$\implies D(x,y) = 6(12 - 12y) - 6^2 = 6(12)(1-y) - 36$$

$$D(0,0) = 36 > 0, q_{xx}(0,0) > 0 \implies q(0,0) \text{ local minimum}$$

$$D(-1,1) = -36 < 0 \implies (-1,1) \text{ is a saddle point}$$

(b) Yes. Charge increases in some directions and decreases in others at a saddle point. This occurs at $(-1,1)$ where the charge is $q(-1,1) = 1$.

(c) No. There is no local maximum of charge on the plate where the charge would decrease in all directions from that local maximum.

(d) Yes. Charge increases in all directions surrounding local minima. This occurs at $(0,0)$ where the charge is $q(0,0) = 0$.

(e) Yes. The charge is a continuous function and the plate is a closed, bounded region. The Extreme Value Theorem guarantees the existence of an absolute maximum and minimum charge on the plate.

■

5. [2350/062824 (26 pts)] The following problems are not related. Justify your answer to all parts using mathematical techniques learned in this Calculus III course.

(a) (8 pts) The mass of a certain object is given by the function $m = \frac{2}{3}\pi l^3 w^{\frac{3}{2}}$ where l is the length of the object and w is its width. Consider two such objects, one (I) with a length of 1 unit and a width of 4 units and another (II) with a length of 4 units and a width of 1 unit. Which object's mass is more sensitive to a small change in length?

(b) (18 pts) A mosquito is buzzing along the path $\mathbf{r}(t) = \frac{t^3}{12}\,\mathbf{i} + \frac{t^2}{2}\,\mathbf{j} + t\,\mathbf{k}$. Length units are meters and time units are minutes. The mosquito is flying in an area where the carbon dioxide concentration, $C(x,y,z)$, in parts per million (ppm), an attractant for mosquitoes, is given by $C(x,y,z) = 3xyz$. Answer the following questions when the mosquito's path passes through the point $\left(\frac{2}{3}, 2, 2\right)$, making sure to include appropriate units.

i. (6 pts) What is the instantaneous rate of change of the carbon dioxide concentration with respect to time?

ii. (6 pts) What is the instantaneous rate of change of the carbon dioxide concentration with respect to distance (arc length)?

iii. (6 pts) What is the greatest rate of change of the carbon dioxide concentration with respect to distance that the mosquito could observe?

SOLUTION:

(a)

$$\mathrm{d}m = \frac{\partial m}{\partial l}\,\mathrm{d}l + \frac{\partial m}{\partial w}\,\mathrm{d}w = 2\pi l^2 w^{3/2}\,\mathrm{d}l + \pi l^3 w^{1/2}\,\mathrm{d}w$$

$$\begin{aligned} \text{object I}: l = 1, w = 4 &\implies \mathrm{d}m = 16\pi\,\mathrm{d}l + 2\pi\,\mathrm{d}w \\ \text{object II}: l = 4, w = 1 &\implies \mathrm{d}m = 32\pi\,\mathrm{d}l + 64\pi\,\mathrm{d}w \rightarrow \text{ more sensitive to small changes in length } l \end{aligned}$$

(b) The mosquito is at the point when $t = 2$.

i.

$$\nabla C(x,y,z) = 3\langle yz, xz, xy\rangle \implies \nabla C\left(\frac{2}{3},2,2\right) = 3\left\langle 4, \frac{4}{3}, \frac{4}{3}\right\rangle = \langle 12,4,4\rangle$$

$$\mathbf{r}'(t) = \left\langle \frac{t^2}{4}, t, 1\right\rangle \implies r'(2) = \langle 1,2,1\rangle$$

$$\left.\frac{\mathrm{d}C}{\mathrm{d}t}\right|_{t=2} = \nabla C\left(\frac{2}{3},2,2\right)\cdot \mathbf{r}'(2) = \langle 12,4,4\rangle \cdot \langle 1,2,1\rangle = 24 \text{ ppm/min}$$

ii.

$$\left.\frac{\mathrm{d}C}{\mathrm{d}s}\right|_{\left(\frac{2}{3},2,2\right)} = \nabla C\left(\frac{2}{3},2,2\right)\cdot \frac{\mathbf{r}'(2)}{\|\mathbf{r}'(2)\|} = \frac{24}{\sqrt{1^2+2^2+1^2}} = \frac{24}{\sqrt{6}} = 4\sqrt{6} \text{ ppm/meter}$$

iii.

$$\left\|\nabla C\left(\frac{2}{3},2,2\right)\right\| = \sqrt{12^2+4^2+4^2} = \sqrt{176} = 4\sqrt{11} \text{ ppm/meter}$$

$\blacksquare$