1. [2350/061424 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) Three nonzero vectors are coplanar if and only if their scalar triple product is 0.

   (b) The rectifying plane for any path of the form $\mathbf{r}(t) = \langle c, f(t), g(t)\rangle$ where $f(t)$ and $g(t)$ are differentiable functions that are not everywhere 0, and c is a real constant, will always be parallel to the yz-plane.

   (c) The magnitude of the torque vector is equivalent to the area of the parallelogram formed by the force and radius vectors.

   (d) The surface given by $y^2 - x^2 + 2x + z^2 + 4z = -3$ is a hyperboloid of two sheets.

   (e) If a force acts at an angle of 60° to the displacement, the work done by the force equals one-half the product of the magnitudes of the force and displacement.

SOLUTION:

(a) **TRUE** If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are nonzero vectors, their scalar triple product is $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ which vanishes if and only if $\mathbf{A}$ is orthogonal to $\mathbf{B} \times \mathbf{C}$, implying $\mathbf{A}$ lies in the plane formed by $\mathbf{B}$ and $\mathbf{C}$.

(b) **FALSE** The curve lies in the plane $x = c$ so its osculating plane is parallel to the yz-plane, not its rectifying plane.

(c) **TRUE** $\|\boldsymbol{\tau}\| = \|\mathbf{r} \times \mathbf{F}\| = \|\mathbf{r}\|\|\mathbf{F}\| \sin\theta =$ area of parallelogram formed by $\mathbf{r}$ and $\mathbf{F}$.

(d) **FALSE** It is a cone.

$$y^2 - x^2 + 2x + z^2 + 4z = -3$$

$$y^2 - (x^2 - 2x + 1 - 1) + z^2 + 4z + 4 - 4 = -3$$

$$y^2 - (x-1)^2 + (z+2)^2 = 0$$

(e) **TRUE** Work $= \mathbf{F} \cdot \mathbf{D} = \|\mathbf{F}\|\|\mathbf{D}\| \cos 60^\circ = \frac{1}{2}\|\mathbf{F}\|\|\mathbf{D}\|$

$\blacksquare$

2. [2350/061424 (24 pts)] Buzz Lightyear is flying along the path $\mathbf{b}(t) = t^2\,\mathbf{i} + (9+t)\,\mathbf{j} + \sqrt{6}t^{3/2}\,\mathbf{k}$. He is being pursued by Emperor Zurg, whose starship is traveling along the path $\mathbf{h}(t) = \dfrac{t^3}{3}\,\mathbf{i} + 4t\,\mathbf{j} + \sqrt{2}t^2\,\mathbf{k}$. Both paths are valid for $t \geq 0$.

   (a) (8 pts) Is there a time, t, when Zurg's ship will intercept Buzz? If so, find it and the point where this occurs. If not, explain why not.

   (b) (16 pts) Zurg's ship only has enough fuel to travel $\frac{32}{3}$ units of distance.

      i. (12 pts) Where is Zurg's ship when it runs out of fuel? Hint: $t^3 + 12t - 32 = 0$ has only one real root which can be found by guessing and checking.

      ii. (4 pts) Is Buzz safe, that is, can Zurg not catch him?

SOLUTION:

(a) We need to see if there exists a t such that $\mathbf{b}(t) = \mathbf{h}(t)$. To this end, we need to solve the following system of equations:

$$t^2 = t^3/3 \tag{1}$$

$$9 + t = 4t \tag{2}$$

$$\sqrt{6}t^{3/2} = \sqrt{2}t^2 \tag{3}$$

Eq. (2) gives $t = 3$, which also satisfies Eqs. (1) and (2). The two paths intersect at the point $\left(9, 12, 9\sqrt{2}\right)$.

(b) i. We need to find the arc length function for Zurg's ship. The ship is at $(0,0,0)$ when $t = 0$. Thus

$$\mathbf{h}'(t) = t^2\,\mathbf{i} + 4\,\mathbf{j} + 2\sqrt{2}t\,\mathbf{k} \implies \|\mathbf{h}'(t)\| = \sqrt{t^4 + 16 + 8t^2} = \sqrt{(t^2+4)^2} = |t^2 + 4| = t^2 + 4$$

$$s(t) = \int_0^t \|\mathbf{h}'(u)\|\,\mathrm{d}u \int_0^t (u^2 + 4)\,\mathrm{d}u = \frac{t^3}{3} + 4t$$

Zurg's ship's fuel runs out after traveling $\frac{32}{3}$ units, or

$$\frac{32}{3} = \frac{t^3}{3} + 4t \implies t^3 + 12t - 32 = 0$$

The only real root of this equation is $t = 2$, so Zurg's ship at this time is at $\left(\frac{8}{3}, 8, 4\sqrt{2}\right)$.

ii. Yes, Buzz is safe. To intercept Buzz, Zurg would have to travel for 3 units of time, but only has fuel for 2.

■

3. [2350/061424 (12 pts)] Find an equation for the surface consisting of all points that are equidistant from the point $(0, -2, 0)$ and the plane $y = 1$. Identify the surface, being as specifc as possible.

SOLUTION:

Let $P(x, y, z)$ be a point on the surface. Then the distance from P to the plane is $|y - 1| = \sqrt{(y-1)^2}$ and from the point P to the given point is $\sqrt{x^2 + (y+2)^2 + z^2}$. Equating these we have

$$\begin{aligned}
\sqrt{(y-1)^2} &= \sqrt{x^2 + (y+2)^2 + z^2} \\
y^2 - 2y + 1 &= x^2 + y^2 + 4y + 4 + z^2 \\
-6y &= x^2 + z^2 + 3 \\
y &= -\frac{1}{6}\left(x^2 + z^2\right) - \frac{1}{2}
\end{aligned}$$

This is a circular paraboloid. ■

4. [2350/061424 (28 pts)] A bird is flying with a velocity vector of $\mathbf{v}(t) = 2\,\mathbf{i} + 2t\,\mathbf{j} + t^2\,\mathbf{k},\ t \geq 0$. When $t = 0$ its position is $\mathbf{r}(0) = 10\,\mathbf{k}$.

(a) (10 pts) Where is the bird when $t = 3$?

(b) (6 pts) Find the curvature of the bird's path.

(c) (6 pts) What is the magnitude of the tangential component of the bird's acceleration?

(d) (6 pts) What is the magnitude of the normal component of the bird's acceleration?

SOLUTION:

(a)

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,\mathrm{d}t = \int \left(2\,\mathbf{i} + 2t\,\mathbf{j} + t^2\,\mathbf{k}\right)\mathrm{d}t = 2t\,\mathbf{i} + t^2\,\mathbf{j} + \frac{t^3}{3}\,\mathbf{k} + \mathbf{C}$$

$$\mathbf{r}(0) = 10\,\mathbf{k} = \mathbf{C}$$

$$\mathbf{r}(t) = 2t\,\mathbf{i} + t^2\,\mathbf{j} + \left(\frac{t^3}{3} + 10\right)\mathbf{k} \implies \mathbf{r}(3) = 6\,\mathbf{i} + 9\,\mathbf{j} + 19\,\mathbf{k}$$

(b) With $\mathbf{r}''(t) = \mathbf{v}'(t)$, we have

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2t & t^2 \\ 0 & 2 & 2t \end{vmatrix} = 2t^2\,\mathbf{i} - 4t\,\mathbf{j} + 4\,\mathbf{k}$$

$$\begin{aligned}
\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| &= \sqrt{4t^4 + 16t^2 + 16} = \sqrt{\left(2t^2 + 4\right)^2} = |2t^2 + 4| = 2t^2 + 4 = 2\left(t^2 + 2\right) \\
\|\mathbf{r}'(t)\| &= \sqrt{4 + 4t^2 + t^4} = \sqrt{\left(t^2 + 2\right)^2} = |t^2 + 2| = t^2 + 2 \\
\kappa(t) &= \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{2\left(t^2 + 2\right)}{\left(t^2 + 2\right)^3} = \frac{2}{\left(t^2 + 2\right)^2}
\end{aligned}$$

(c)

$$a_T(t) = \frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}(t)\| = \frac{\mathrm{d}}{\mathrm{d}t}\left(t^2 + 2\right) = 2t = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\|\mathbf{r}'(t)\|}$$

(d)

$$a_N(t) = \sqrt{\|\mathbf{a}(t)\|^2 - \left[a_T(t)\right]^2} = \sqrt{4 + 4t^2 - 4t^2} = 2 = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}$$

■

5. [2350/061424 (26 pts)] Consider the points $A(1,0,1),\ B(2,2,1),\ C(1,2,2)$ and the line $\dfrac{1-x}{2} = 1-y = \dfrac{z-2}{3}$.

(a) (8 pts) Find the equation of the plane containing the points, writing your answer in the form $ax+by+cz=d$.

(b) (6 pts) Find the area of the triangle whose vertices are the points A, B, C.

(c) (6 pts) Where does the line intersect the plane?

(d) (6 pts) Find the angle at which the line hits the plane.

SOLUTION:

(a) Two vectors in the plane are

$$\overrightarrow{AB} = \langle 2,2,1\rangle - \langle 1,0,1\rangle = \langle 1,2,0\rangle$$
$$\overrightarrow{AC} = \langle 1,2,2\rangle - \langle 1,0,1\rangle = \langle 0,2,1\rangle$$

A normal vector to the plane is

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ 0 & 2 & 1 \end{vmatrix} = 2\,\mathbf{i} - \mathbf{j} + 2\,\mathbf{k}$$

We then have $2(x-1) - (y-0) + 2(z-1) = 0 \implies 2x - y + 2z = 4$ as the plane's equation.

(b)

$$\text{Area of } \triangle ABC = \frac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \frac{1}{2}\sqrt{2^2 + (-1)^2 + 2^2} = \frac{3}{2}$$

(c) The parametric equations of the line are $x = 1-2t,\ y = 1-t,\ z = 2+3t$. Putting these into the equation of the plane yields

$$2(1-2t) - (1-t) + 2(2+3t) = 4 \implies 5 + 3t = 4 \implies t = -\frac{1}{3} \implies \text{intersection point is } (x,y,z) = \left(\frac{5}{3}, \frac{4}{3}, 1\right)$$

(d) The direction vector of the line is $\mathbf{v} = \langle -2,-1,3\rangle$ and the angle between this vector and the normal to the plane, $\mathbf{n} = \langle 2,-1,2\rangle$, is

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}\cdot\mathbf{v}}{\|\mathbf{n}\|\|\mathbf{v}\|}\right) = \cos^{-1}\left(\frac{(2)(-2) + (-1)(-1) + (2)(3)}{\sqrt{2^2 + (-1)^2 + 2^2}\sqrt{(-2)^2 + (-1)^2 + 3^2}}\right) = \cos^{-1}\left(\frac{3}{3\sqrt{14}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right)$$

The angle the line makes with the plane is then $\dfrac{\pi}{2} - \cos^{-1}\left(\dfrac{1}{\sqrt{14}}\right)$.

■