- This exam is worth 150 points and has 6 problems.
- **Show all work and simplify your answers!** Answers with no justification will receive no points unless otherwise noted.
- **Please begin each problem on a new page.**
- **DO NOT** leave the exam until you have satisfactorily scanned and uploaded your exam to Gradescope.
- You are taking this exam in a proctored and honor code enforced environment. **NO** calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5"× 11" crib sheet with writing on two sides.
- Remote students are allowed use of a computer during the exam only for a live video of their hands and face and to view the exam in the Zoom meeting.

0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2350/072823 (20 pts)] A jellyfish swims along the path $\mathbf{r}(t) = t^2\,\mathbf{i} + \mathbf{j} - e^t\,\mathbf{k}$ from $0 \le t \le 3$ catching plankton as it moves along. If the density of plankton in the water is given by $P(x, y, z) = y\sqrt{4x + z^2}$ g/m find the total amount of plankton the jellyfish caught. Include units in your final answer.

SOLUTION:

$$\mathbf{r}'(t) = \langle 2t, 0, -e^t\rangle \implies \|\mathbf{r}'(t)\| = \sqrt{4t^2 + e^{2t}}$$

$$P(\mathbf{r}(t)) = \sqrt{4t^2 + e^{2t}}$$

$$\begin{aligned}
\text{Total plankton} &= \int_{\mathcal{C}} P(x, y, z)\,\mathrm{d}s = \int_0^3 P(\mathbf{r}(t))\|\mathbf{r}'(t)\|\,\mathrm{d}t \\
&= \int_0^3 \sqrt{4t^2 + e^{2t}}\sqrt{4t^2 + e^{2t}}\,\mathrm{d}t = \int_0^3 \left(4t^2 + e^{2t}\right)\mathrm{d}t \\
&= \left(\frac{4}{3}t^3 + \frac{1}{2}e^{2t}\right)\Bigg|_0^3 = \frac{4}{3}(3^3 - 0^3) + \frac{1}{2}(e^6 - 1) \\
&= 36 + \frac{e^6 - 1}{2} \text{ grams}
\end{aligned}$$

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2. [2350/072823 (30 pts)] The force of the current in Penelope the platypus's river can be described by the vector field $\mathbf{F} = (2x + \tan y)\,\mathbf{i} + (x \sec^2 y)\,\mathbf{j}$. Penelope swims first along the curve $\mathcal{C}_1$ to the base of a waterfall then later returns along the curve $\mathcal{C}_2$ with

- $\mathcal{C}_1$: the line segment from $(2, 1)$ to $(2, -1)$
- $\mathcal{C}_2$: the curve $x = 2 + \cos\left(\frac{\pi}{2}y\right)$ from $(2, -1)$ to $(2, 1)$

(a) (10 pts) Directly calculate the work done by the current on Penelope along $\mathcal{C}_1$ by evaluating an appropriate line integral.

(b) (10 pts) Find the potential function of $\mathbf{F}$.

(c) (5 pts) Using a theorem from Calculus 3, determine the work done by the current on Penelope along $\mathcal{C}_2$.

(d) (5 pts) Determine the total work done by the current along the union of the two paths: $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$.

SOLUTION:

(a) We start by parameterizing the curve. One option is to use the parameterization

$$\mathbf{r}(t) = (1 - t)\langle 2, 1\rangle + t\langle 2, -1\rangle = \langle 2, 1 - 2t\rangle,\ 0 \le t \le 1$$

$$\mathbf{F}(\mathbf{r}(t)) = \left\langle 4 + \tan(1 - 2t), 2\sec^2(1 - 2t)\right\rangle$$

$$\mathbf{r}'(t) = \langle 0, -2\rangle$$

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \langle 4 + \tan(1 - 2t), 2\sec^2(1 - 2t)\rangle \cdot \langle 0, -2\rangle = -4\sec^2(1 - 2t)$$

Then

$$\begin{aligned}
\text{Work} &= \int_{\mathcal{C}_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t \\
&= \int_0^1 -4\sec^2(1-2t)\,\mathrm{d}t = 2\tan(1-2t)\Big|_0^1 \\
&= 2\left[\tan(-1) - \tan 1\right] \\
&= -4\tan 1
\end{aligned}$$

(b) We seek the potential function f, such that $\nabla f = \langle f_x, f_y \rangle = \mathbf{F}$.

$$f(x,y) = \int f_x\,\mathrm{d}x = \int (2x + \tan y)\,\mathrm{d}x = x^2 + x\tan y + g(y)$$

$$f_y(x,y) = x\sec^2 y + g'(y) = x\sec^2 y \implies g'(y) = 0 \implies g(y) = c$$

$$f(x,y) = x^2 + x\tan y + c$$

(c) $\mathbf{F} = \nabla f$ is continuous on both curves. Thus, we can apply the Fundamental Theorem of Line Integrals to find that the work done by the current along $\mathcal{C}_2$ is simply

$$\text{Work} = f(2,1) - f(2,-1) = (2^2 + 2\tan 1) - \left[2^2 + 2\tan(-1)\right] = 2\tan 1 - (-2\tan 1) = 4\tan 1$$

One could also conclude that the work is independent of the path so that $\displaystyle\int_{\mathcal{C}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{-\mathcal{C}_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int_{\mathcal{C}_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 4\tan 1.$

(d)

$$\text{Work} = \int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}_1 \cup \mathcal{C}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{\mathcal{C}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -4\tan 1 + 4\tan 1 = 0$$

This conclusion could have also be drawn from the fact that the path $\mathcal{C}_1 \cup \mathcal{C}_2$ is a closed loop in a conservative vector field, implying that the work done on the path is zero.

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3. [2350/072823 (36 pts)] Consider the open surface cut from $z = -3\sqrt{x^2 + y^2}$ where $-3 \le z \le 0$.

(a) (5 pts) What quadric surface is this?

(b) (10 pts) Give a parameterization of the boundary curve, $\mathcal{C}$, of this surface with a counterclockwise orientation when viewed from above.

(c) (5 pts) What portion of a plane shares the same boundary?

(d) (16 pts) Use Stokes' theorem to evaluate $\displaystyle\int_{\mathcal{C}} -3yz\,\mathrm{d}x + 7x\,\mathrm{d}y + z\,\mathrm{d}z.$

SOLUTION:

(a) It is the bottom portion of a cone with vertex at the origin and height of 3 units.

(b) One option is $\mathbf{r}(t) = \langle \cos t, \sin t, -3 \rangle$ with $0 \le t \le 2\pi$.

(c) The portion of the plane $z = -3$ within $x^2 + y^2 = 1$

(d) Two surfaces share this boundary curve, the plane $z = -3$ and the cone. We'll use the plane $(\mathcal{S})$ since it make things simpler. Thus, $g(x,y,z) = z, \nabla g = \mathbf{k}$, which gives the proper orientation of the surface given the orientation of the boundary curve. Projecting the surface onto the xy-plane gives $\mathbf{p} = \mathbf{k}, |\nabla g \cdot \mathbf{p}| = 1$ and the integration region $\mathcal{R}$ is the unit disk $x^2 + y^2 \le 1$. From the given integral, the vector field is $\mathbf{F} = \langle -3yz, 7x, z \rangle$. Thus

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -3yz & 7x & z \end{vmatrix} = \langle 0, -3y, 3z + 7 \rangle$$

$$\int_{\mathcal{C}} -3yz\,\mathrm{d}x + 7x\,\mathrm{d}y + z\,\mathrm{d}z = \iint_{\mathcal{S}} \nabla \times \langle -3yz, 7x, z\rangle \cdot \mathbf{n}\,\mathrm{d}S$$
$$= \iint_{\mathcal{R}} (3z+7)\,\mathrm{d}A = \iint_{\mathcal{R}} [3(-3)+7]\,\mathrm{d}A = \iint_{\mathcal{R}} -2\,\mathrm{d}A$$
$$= -2\,\mathrm{area}(\mathcal{R}) = -2\pi(1)^2 = -2\pi$$

4. [2350/072823 (18 pts)] Consider the closed boundary $\mathcal{C}$ made by the curves $x = y^2$ and $x = -y^2 + 2$ oriented counterclockwise. Compute the flux of $\mathbf{H}$ through $\mathcal{C}$ if $\mathbf{H} = \langle e^{-y^2} + 3x^2, \ln x + 6\rangle$.

SOLUTION:

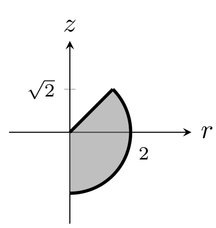
Evaluating the integral directly will not be possible due to the presence of e^{-y^2} term. Instead, we use Green's theorem, letting $\mathcal{D}$ be the region enclosed by the curve, $\mathcal{C}$.

$$\text{Flux} = \int_{\mathcal{C}} \mathbf{H}\cdot\mathbf{n}\,\mathrm{d}s = \iint_{\mathcal{D}} \left[\frac{\partial}{\partial x}\left(e^{-y^2} + 3x^2\right) + \frac{\partial}{\partial y}(\ln x + 6)\right]\mathrm{d}A$$
$$= \int_{-1}^{1}\int_{y^2}^{-y^2+2} 6x\,\mathrm{d}x\,\mathrm{d}y = 3\int_{-1}^{1} x^2\Big|_{y^2}^{-y^2+2}\,\mathrm{d}A$$
$$= 12\int_{-1}^{1}(-y^2+1)\,\mathrm{d}y = 12\left(-\frac{y^3}{3} + y\right)\Bigg|_{-1}^{1}$$
$$= 16$$

5. [2350/072823 (22 pts)] Consider a three dimensional solid, $\mathcal{E}$, bounded within $\mathcal{S}_1 : x^2 + y^2 + z^2 = 4$ and below $\mathcal{S}_2 : z = \sqrt{x^2 + y^2}$. Find the outward flux of $\mathbf{F} = \left\langle \sin y, x\ln(z+1), z^2\right\rangle$ through the boundary of $\mathcal{E}$.

SOLUTION:

Sketch of the solid in the rz-plane.



We utilize the Divergence theorem.

$$\iint_{\partial\mathcal{E}} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \iiint_{\mathcal{E}} \nabla\cdot\mathbf{F}\,\mathrm{d}V$$
$$= \iiint_{\mathcal{E}} 2z\,\mathrm{d}V = 2\int_0^{2\pi}\int_{\pi/4}^{\pi}\int_0^2 (\rho\cos\phi)\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$
$$= 2(2\pi)\int_{\pi/4}^{\pi}\int_0^2 \rho^3\cos\phi\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi = 2\pi\int_{\pi/4}^{\pi}\sin 2\phi\left(\frac{1}{4}\rho^4\right)\Big|_0^2\,\mathrm{d}\phi$$
$$= 8\pi\int_{\pi/4}^{\pi}\sin 2\phi\,\mathrm{d}\phi = 4\pi(-\cos 2\phi)\Big|_{\pi/4}^{\pi}$$
$$= -4\pi$$

Note that cylindrical coordinates could also be used, yielding:

$$\begin{aligned}\iint_{\partial\mathcal{E}} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S &= \int_0^{2\pi}\int_0^{\sqrt{2}}\int_{-\sqrt{4-r^2}}^{r} 2z\,r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta + \int_0^{2\pi}\int_{\sqrt{2}}^{2}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} 2z\,r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta \\ &= 2\pi\int_0^{\sqrt{2}} \left[r^2 - \left(4-r^2\right)\right] r\,\mathrm{d}r + 0 \\ &= 4\pi\int_0^{\sqrt{2}} \left(r^3 - 2r\right)\mathrm{d}r = 4\pi\left(\frac{r^4}{4} - r^2\right)\Bigg|_0^{\sqrt{2}} = -4\pi\end{aligned}$$

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6. [2350/072823 (24 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

(a) The function $f(x,y) = 1 - x^2 - y^2$ is guaranteed to have a minimum value for all x, y in the first octant.

(b) The function $g(x,y) = e^{xy}$ has a saddle point at the origin.

(c) The cross product of $\mathbf{k}$ and the acceleration vector of the path $\mathbf{r}(t) = 2t\,\mathbf{i} + 5t\,\mathbf{j} + t^2\,\mathbf{k}$ is never the zero vector.

(d) The line whose vector equation is $\mathbf{r}(t) = \langle 1, 6, 2\rangle + t\langle -1, -3, 2\rangle$ intersects the xz-plane at the point $(-1, 0, 6)$.

(e) $\displaystyle\lim_{(x,y)\to(-1,0)} \frac{xy + x}{(x+1)^2 + y^2 - 2}$ does not exist.

(f) The instantaneous rate of change of the function $g(x,y) = 4x^2 + 2x - 3y^2$ at the origin is largest in the $+\mathbf{j}$ direction.

SOLUTION:

(a) **FALSE** The function is continuous but the first octant is neither closed nor bounded so no conclusions can be drawn from the Extreme Value Theorem.

(b) **TRUE**

$$f_x = ye^{xy},\ f_{xx} = y^2e^{xy}$$
$$f_y = xe^{xy},\ f_{yy} = x^2e^{xy}$$
$$f_{xy} = e^{xy}(1 + xy)$$

The only critical point is the origin, $(0,0)$. At that point, $f_{xx}(0,0) = f_{yy}(0,0) = 0$ and $f_{xy}(0,0) = 1$ so that $D(0,0) = -1 < 0$, implying that the origin is a saddle point.

(c) **FALSE** It is always the zero vector.

$$\mathbf{v} = \mathbf{r}'(t) = 2\,\mathbf{i} + 5\,\mathbf{j} + 2t\,\mathbf{k}$$
$$\mathbf{a} = \mathbf{r}''(t) = 2\,\mathbf{k}$$
$$\mathbf{k}\times\mathbf{a} = \mathbf{k}\times(2\,\mathbf{k}) = 2\,(\mathbf{k}\times\mathbf{k}) = \mathbf{0}$$

(d) **TRUE** Points in the xz-plane have y-coordinate of 0. The y-component of the line is $y = 6 - 3t$ which vanishes if $t = 2$. In this case, $x(2) = 1 - 1(2) = -1$ and $z(2) = 2 + 2(2) = 6$.

(e) **FALSE** Use direct substitution.

$$\lim_{(x,y)\to(-1,0)} \frac{xy + x}{(x+1)^2 + y^2 - 2} = \frac{(-1)(0) - 1}{(-1+1)^2 + 0^2 - 2} = \frac{1}{2}$$

(f) **FALSE** It is largest in the $+\mathbf{i}$ direction. The maximum instantaneous rate of change occurs in the direction of the gradient:

$$\nabla g(x,y) = (8x + 2)\,\mathbf{i} - 6y\,\mathbf{j} \implies \nabla g(0,0) = 2\,\mathbf{i}$$

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