1. [2350/071423 (20 pts)] The portion of river, $\mathcal{D}$, that Penelope the platypus inhabits is bounded by the functions $x+y+4=0$, $x+y+4=8$, $x-2y=-1$, and $x-2y=1$. The depth (meters) of Penelope's river is described by $g(x,y)=3y+4$. Use a change of variables to find the volume of Penelope's river in the region $\mathcal{D}$. Include appropriate units in your answer.

SOLUTION:

One change of variables is $u=x+y+4$ and $v=x-2y$. This gives the new region of integration as $0\le u\le 8$ and $-1\le v\le 1$. Rearranging we find that $y=\frac{1}{3}(u-v-4)$ and $x=\frac{1}{3}(2u+v-8)$. Thus,

$$J(u,v)=\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}=\begin{vmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{vmatrix}=-\frac{1}{3}$$

$$g(u,v)=3\left[\frac{1}{3}(u-v-4)\right]+4=u-v$$

$$\text{Volume}=\iint_{\mathcal{D}}(3y+4)\,\mathrm{d}A=\int_{-1}^{1}\int_{0}^{8}(u-v)\left|-\frac{1}{3}\right|\mathrm{d}u\,\mathrm{d}v$$

$$=\frac{1}{3}\int_{-1}^{1}\left(\frac{u^2}{2}-uv\right)\Big|_0^8\,\mathrm{d}v$$

$$=\frac{1}{3}\int_{-1}^{1}(32-8v)\,\mathrm{d}v$$

$$=\frac{1}{3}\left(32v-4v^2\right)\Big|_{-1}^{1}=\frac{64}{3}\ \mathrm{m}^3$$

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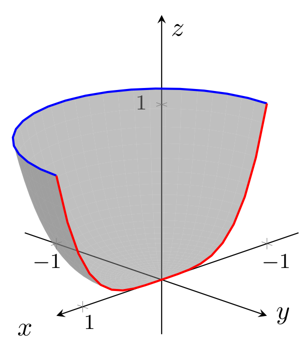
2. [2350/071423 (27 pts)] During a tea party with the Mad Hatter, you are given exactly a 1/2 teacup of tea (pictured below) where the surface of the teacup is given by the equation $z=(x^2+y^2)^2$. It lies in the region where $y\le 0$ and $0\le z\le 1$. You wish to know the volume of tea you received. For each part below, set up the triple integral using the given coordinate system and order of integration. **DO NOT EVALUATE** the triple integrals. Note: To receive full credit your bounds **must** match the solid as it is pictured (so study the figure closely).

(a) Cartesian Coordinates with order $\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x$.

(b) Cylindrical Coordinates with order $\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$

(c) Spherical coordinates with order $\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$



SOLUTION:

(a)

$$\int_{-1}^{1}\int_{x^4}^{1}\int_{-\sqrt{\sqrt{z}-x^2}}^{0}\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x$$

(b)

$$\int_{\pi}^{2\pi}\int_{0}^{1}\int_{r^4}^{1}r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta \quad \text{or} \quad \int_{-\pi}^{0}\int_{0}^{1}\int_{r^4}^{1}r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

(c)

$$\int_{\pi}^{2\pi}\int_{0}^{\pi/4}\int_{0}^{\sec\phi}\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta+\int_{\pi}^{2\pi}\int_{\pi/4}^{\pi/2}\int_{0}^{\left(\cos\phi\csc^4\phi\right)^{1/3}}\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

or

$$\int_{-\pi}^{0}\int_{0}^{\pi/4}\int_{0}^{\sec\phi}\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta+\int_{-\pi}^{0}\int_{\pi/4}^{\pi/2}\int_{0}^{\left(\cos\phi\csc^4\phi\right)^{1/3}}\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

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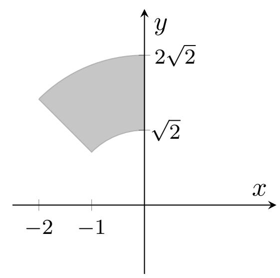
3. [2350/071423 (18 pts)] Rewrite the following integrals as a single integral in polar coordinates **and** evaluate it. Drawing a picture should be beneficial.

$$\int_{-2}^{-1}\int_{-x}^{\sqrt{8-x^2}}3x\,\mathrm{d}y\,\mathrm{d}x+\int_{-1}^{0}\int_{\sqrt{2-x^2}}^{\sqrt{8-x^2}}3x\,\mathrm{d}y\,\mathrm{d}x$$

SOLUTION:



This is a portion of an annulus with inner radius of $\sqrt{2}$ and outer radius of $2\sqrt{2}$ in the second quadrant. Rewriting in polar coordinates we have:

$$\int_{\pi/2}^{3\pi/4}\int_{\sqrt{2}}^{2\sqrt{2}}3r^2\cos\theta\,\mathrm{d}r\,\mathrm{d}\theta=\left(r^3\Big|_{\sqrt{2}}^{2\sqrt{2}}\right)\left(\sin\theta\Big|_{\pi/2}^{3\pi/4}\right)=\left(16\sqrt{2}-2\sqrt{2}\right)\left(\frac{\sqrt{2}}{2}-1\right)=14\left(1-\sqrt{2}\right)$$

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4. [2350/071423 (22 pts)] Consider a thin leaf whose shape can be described by the region bounded by the curves $y=x^2$ and $y=x^4$ with $0\le x\le 1$.

   (a) (16 pts)] Assume the mass of the leaf is $\frac{1}{20}$ grams and its mass density is $\rho(x,y)=x\sqrt{y}$ grams per square centimeter.

      i. (6 pts)] Find the moment of the leaf about the y-axis using the integration order $\underline{\mathrm{d}y\,\mathrm{d}x}$.

      ii. (6 pts)] Find the moment of the leaf about the x-axis using the integration order $\underline{\mathrm{d}x\,\mathrm{d}y}$.

      iii. (4 pts)] What is center of mass, $(\bar{x},\bar{y})$, of the leaf?

   (b) (6 pts)] Now suppose the power density of sunlight hitting the leaf is $P(x,y)=k(1-x^2)$ watts per square centimeter and you know that the total power hitting the leaf is 8 watts. What is k?

SOLUTION:

(a) i.

$$M_y=\iint\limits_{\text{leaf}}x\rho(x,y)\,\mathrm{d}A=\int_0^1\int_{x^4}^{x^2}x^2y^{1/2}\,\mathrm{d}y\,\mathrm{d}x$$

$$=\int_0^1x^2\left(\tfrac{2}{3}y^{3/2}\Big|_{x^4}^{x^2}\right)\mathrm{d}x=\frac{2}{3}\int_0^1x^2\left(x^3-x^6\right)\mathrm{d}x$$

$$=\frac{2}{3}\left(\frac{x^6}{6}-\frac{x^9}{9}\right)\Bigg|_0^1=\frac{2}{3}\left(\frac{9-6}{54}\right)=\frac{1}{27}$$

ii.

$$M_x = \iint\limits_{\text{leaf}} y\rho(x,y)\,\mathrm{d}A = \int_0^1 \int_{y^{1/2}}^{y^{1/4}} xy^{3/2}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^1 y^{3/2}\left(\tfrac{1}{2}x^2\Big|_{y^{1/2}}^{y^{1/4}}\right)\mathrm{d}y = \frac{1}{2}\int_0^1 y^{3/2}\left(y^{1/2}-y\right)\mathrm{d}y$$

$$= \frac{1}{2}\left(\frac{y^3}{3} - \frac{2y^{7/2}}{7}\right)\Bigg|_0^1 = \frac{1}{2}\left(\frac{7-6}{21}\right) = \frac{1}{42}$$

iii.

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{1/27}{1/20}, \frac{1/42}{1/20}\right) = \left(\frac{20}{27}, \frac{10}{21}\right)$$

(b) The total power is equal to the double integral of the power density.

$$8 = \int_0^1 \int_{x^4}^{x^2} k\left(1-x^2\right)\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^1 k(1-x^2)y\Big|_{x^4}^{x^2}\mathrm{d}x = \int_0^1 k\left(1-x^2\right)\left(x^2-x^4\right)\mathrm{d}x$$

$$= k\int_0^1 \left(x^2 - 2x^4 + x^6\right)\mathrm{d}x = k\left(\frac{x^3}{3} - \frac{2x^5}{5} + \frac{x^7}{7}\right)\Bigg|_0^1$$

$$= k\left(\frac{35-42+15}{105}\right) = \frac{8k}{105} \implies k = 105$$

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5. [2350/071423 (13 pts)] Use spherical coordinates to evaluate $\iiint_{\mathcal{E}} z\,\mathrm{d}V$ where $\mathcal{E}$ is the solid that lies within the sphere $x^2+y^2+z^2=4$, above the xy-plane, below the cone $z=\sqrt{x^2+y^2}$ and between the planes $y=\frac{\sqrt{3}}{3}x$ and $y=\sqrt{3}x$.

SOLUTION:

In spherical coordinates the sphere is given by $\rho=2$, the cone by $\phi=\frac{\pi}{4}$ and the planes by $\theta=\frac{\pi}{6}$ and $\theta=\frac{\pi}{3}$. Then

$$\iiint_{\mathcal{E}} z\,\mathrm{d}V = \int_{\pi/6}^{\pi/3}\int_{\pi/4}^{\pi/2}\int_0^2 \rho\cos\phi\left(\rho^2\sin\phi\right)\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

$$= \left(\int_{\pi/6}^{\pi/3}\mathrm{d}\theta\right)\left(\int_0^2 \rho^3\,\mathrm{d}\rho\right)\left(\int_{\pi/4}^{\pi/2}\sin\phi\cos\phi\,\mathrm{d}\phi\right)$$

$$= \frac{\pi}{6}\left(\frac{\rho^4}{4}\Big|_0^2\right)\left(\frac{1}{2}\int_{\pi/4}^{\pi/2}\sin 2\phi\,\mathrm{d}\phi\right)$$

$$= \frac{\pi}{6}(4)\left(-\frac{1}{4}\cos 2\phi\Big|_{\pi/4}^{\pi/2}\right)$$

$$= \frac{\pi}{6}$$

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