1. [2350/063023 (30 pts)] Penelope the platypus adventures about on the surface of a river, where distances are measured in meters. Suppose the temperature (°C) on the river's surface is given by

$$T(x, y) = -(2x^2 - 4)^2 + 4y^2$$

(a) [10 pts] Penelope's current location is at $(1, 3)$. Platypus risk overheating in Australian summers and can only cool down by immersing themselves in cold water.

i. [5 pts] Find the direction Penelope should travel to reach lower temperatures the fastest.

ii. [5 pts] What is the instantaneous rate of temperature change with respect to distance $(\mathrm{d}T/\mathrm{d}s)$ in this direction?

(b) [20 pts] Penelope is now at the point $(2, -2)$, a location with a comfortable temperature.

i. [6 pts] What is the equation of the level curve for this temperature? Sketch it.

ii. [6 pts] What direction(s) should Penelope travel from this point to remain at this temperature? Report your answer(s) as unit vectors.

iii. [8 pts] If Penelope were swimming along the path $\mathbf{r}(t) = (t+2)\,\mathbf{i} + \big(t^2 + 2t - 2\big)\,\mathbf{j}$, what is the instantaneous rate of change of temperature with respect to time (seconds) $(\mathrm{d}T/\mathrm{d}t)$ when she reaches the point $(2, -2)$?

SOLUTION:

(a) i. The direction of maximum decrease in the temperature is $-\nabla T$.

$$\nabla T(x, y) = \langle -8x(2x^2 - 4), 8y\rangle \implies \nabla T(1, 3) = \langle -8(-2), 24\rangle = \langle 16, 24\rangle$$

Thus Penelope should travel along $-\nabla T(1, 3) = \langle -16, -24\rangle$.

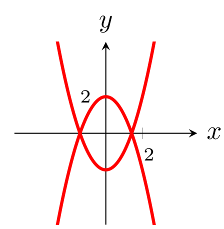
ii. The rate of temperature change with respect to distance in this direction is

$$\left.\frac{\mathrm{d}T}{\mathrm{d}s}\right|_{(1,3)} = \nabla T(1, 3) \cdot \frac{-\nabla T(1, 3)}{\|-\nabla T(1, 3)\|} = -\|-\nabla T(1, 3)\|$$

$$= -\sqrt{16^2 + 24^2} = -\sqrt{832} = -\sqrt{8^2\,(2^2 + 3^2)} = -8\sqrt{13}\ \text{°C/m}$$

(b) i. $T(2, -2) = -(8-4)^2 + 4(4) = 0$ °C. Then the level curve is

$$0 = -(2x^2 - 4)^2 + 4y^2 \implies (2x^2 - 4)^2 = 4y^2 \implies y = \pm\frac{1}{2}(2x^2 - 4) \implies y = \pm(x^2 - 2)$$



ii. We need to find unit vector(s), $\mathbf{u} = \langle u_1, u_2\rangle$, such that the directional derivative is 0.

$$D_{\mathbf{u}}T(2, -2) = \nabla T(2, -2) \cdot \langle u_1, u_2\rangle = 0$$

$$\langle -16(4), -16\rangle \cdot \langle u_1, u_2\rangle = 0$$

$$-64u_1 - 16u_2 = 0$$

$$4u_1 + u_2 = 0$$

One option satisfying this equation is $u_1 = 1, u_2 = -4$. Thus Penelope could move in the direction $\pm\left\langle \frac{1}{\sqrt{17}}, \frac{-4}{\sqrt{17}}\right\rangle$ to remain at the comfortable temperature.

iii. Penelope will reach the comfortable spot when $t = 0$.

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\partial T}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial T}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = \nabla T \cdot \mathbf{r}'(t) = \langle -8x(2x^2 - 4), 8y\rangle \cdot \langle 1, 2t + 2\rangle$$

$$\left.\frac{\mathrm{d}T}{\mathrm{d}t}\right|_{t=0} = \langle -64, -16\rangle \cdot \langle 1, 2\rangle = -96\ \text{°C/sec}$$

2. [2350/063023 (23 pts)] The following parts (a) and (b) are not related.

(a) [15 pts] Consider

$$f(x,y) = \frac{y^2x^4 - 4x^4y + 4x^4}{y-2}$$

i. [5 pts] What are the domain and range of $f(x,y)$?

ii. [5 pts] Find $\lim_{(x,y)\to(-2,2)} f(x,y)$ or show that it does not exist.

iii. [5 pts] Is $f(x,y)$ continuous on $\mathbb{R}^2$? Justify your answer.

(b) [8 pts] Find an equation of the tangent plane to the surface $xy + yz + xz = 5$ at the point $(1,2,1)$. Write your answer in the form $ax + by + cz = d$.

SOLUTION:

(a) i. The domain is $\left\{(x,y) \in \mathbb{R}^2 \,\middle|\, y \neq 2\right\}$. The range is $\mathbb{R}$ since f is unbounded (it increases/decreases without bound as $y \to 2$).

ii. Direct substitution yields the indeterminate form $\frac{0}{0}$. Our next approach is to try algebraic manipulation.

$$\begin{aligned} f(x,y) &= \frac{y^2x^4 - 4x^4y + 4x^4}{y-2} \\ &= \frac{x^4(y^2 - 4y + 4)}{y-2} \\ &= \frac{x^4(y-2)^2}{y-2} \\ &= x^4(y-2) \end{aligned}$$

Thus $\lim_{(x,y)\to(-2,2)} \frac{y^2x^4 - 4x^4y + 4x^4}{y-2} = \lim_{(x,y)\to(-2,2)} x^4(y-2) = (-2)^4(0) = 0$

iii. No. The function is not defined when $y = 2$ and thus is not continuous when $y = 2$. Therefore, it is not continuous on $\mathbb{R}^2$. (Remark: Note that $\lim_{(x,y)\to(x_0,2)} f(x,y) = 0$ for all x_0)

(b) The given surface is a level surface of the function $F(x,y,z) = xy + yz + xz$. A normal to the given surface is ∇F and, with the given point, will provide the necessary information to find the equation of the tangent plane.

$$\nabla F = \langle F_x, F_y, F_z \rangle = \langle y+z, x+z, y+x \rangle$$

$$\nabla F(1,2,1) = \langle 3,2,3 \rangle$$

$$3(x-1) + 2(y-2) + 3(z-1) = 0$$

$$3x + 2y + 3z = 10$$

3. [2350/063023 (31 pts)] You and your friend Chaplin are on a quest through a forest in search of a magical tome that can solve any math problem for you. A sorceress gave you a clue on its location: it is at the highest point on or within the boundary $3x^2 + y^2 = 9$. The elevation in that region is given by

$$f(x,y) = -4(x-1)^2 - y^2 + 100$$

(a) [5 pts] Chaplin questions whether there even is a highest point in this region. Is there any guarantee that you can offer him that there is a highest point in this region? (No calculations please, just write a sentence with any relevant facts/theorems that would convince Chaplin.)

(b) [10 pts] Find and classify all critical points within the boundary.

(c) [12 pts] Using Lagrange Multipliers, determine if there are any extrema on the boundary.

(d) [4 pts] Based on the work done above, report the location of the magical item and the elevation at that point.

SOLUTION:

(a) The function $f(x,y)$ describing the elevation is continuous inside and on the given boundary which comprises a closed, bounded set. Thus the Extreme Value Theorem guarantees that a maximum elevation does exist.

(b)

$$f_x = -8(x-1) \qquad f_y = -2y$$
$$0 = f_x = -8(x-1) \implies x = 1$$
$$0 = f_y = -2y \implies y = 0$$

Thus the only critical point is $(1, 0)$. To classify the critical point, we use the Second Derivatives test.

$$f_{xx} = -8, \quad f_{yy} = -2, \quad f_{xy} = 0$$
$$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - f_{xy}^2(x, y) = -8(-2) - 0^2 = 16$$

$D(1, 0) = 16 > 0$ and $f_{xx} = -8 < 0$ which implies that $f(1, 0) = 100$ is a local maximum.

(c) The constraint for the Lagrange multiplier method is $g(x, y) = 3x^2 + y^2 = 9$.

$$\nabla f = \lambda \nabla g \implies \langle -8(x-1), -2y \rangle = \lambda \langle 6x, 2y \rangle$$

$$-8(x-1) = \lambda 6x \quad (1)$$
$$-2y = \lambda 2y \quad (2)$$
$$3x^2 + y^2 = 9 \quad (3)$$

Equation (2) can be written as $y(\lambda + 1) = 0 \implies y = 0$ or $\lambda = -1$. In the case $y = 0$, $x = \pm\sqrt{3}$ from Eq. (3). In the case $\lambda = -1$, $x = 4$ from Eq. (1) which implies $y^2 = 9 - 3(16)$ in Eq. (3) which is impossible. Thus the only two critical points on the boundary are $\left(\sqrt{3}, 0\right)$ and $\left(-\sqrt{3}, 0\right)$. $f\left(\sqrt{3}, 0\right) = -4\left(\sqrt{3} - 1\right)^2 + 100$ is a maximum on the boundary and $f\left(-\sqrt{3}, 0\right) = -4\left(-\sqrt{3} - 1\right)^2 + 100$ is a minimum on the boundary.

(d) The absolute maximum in the region (the location of the magical tome) is at $(1, 0)$ with an elevation 100.

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4. [2350/063023 (16 pts)] The following problems are not related.

(a) [6 pts] Suppose you found the second order Taylor approximation, $T_2(x, y)$, centered at $(1, 2)$, of a function $g(x, y)$. You also know that throughout $\mathbb{R}^2$

$$|g_{xx}| \leq 4 \qquad |g_{xy}| \leq e^2 \qquad |g_{yy|} \leq 1$$
$$|g_{xxx}| \leq 3 \qquad |g_{xxy}| \leq 1 \qquad |g_{yyx}| \leq 7 \qquad |g_{yyy}| \leq 5$$

You want to use $T_2(x, y)$ to estimate the value of $g(x, y)$ when $-3 \leq x - 1 \leq 3$ and $-0.2 \leq y - 2 \leq 0.2$. What is the maximum error you can expect the approximation to contain?

(b) [10 pts] The volume of the frustum of a right circular cone is given by $V(r, R, h) = \frac{\pi}{3}h\left(R^2 + Rr + r^2\right)$, where h is the frustum's height, R is the radius of its base, and r is the radius of its top. The measurements of the frustum are $r = 1, R = 2, h = 3$ inch with a possible error of 0.01 inch in the radius measurements and 0.03 inch in the height measurement. Use differentials to estimate the possible error in the computed volume.

SOLUTION:

(a) For $T_2(x, y)$, we need to bound the third order derivatives of $g(x, y)$ on the rectangle, R, given by $|x - 1| < 3, |y - 2| < 0.2$; that is, we need

$$M = \max_{(x,y)\in R} \{|g_{xxx}|, |g_{xxy}|, |g_{yyx}|, |g_{yyy}|\} = \max_{(x,y)\in R} \{3, 1, 7, 5\} = 7$$

Then, using the error formula we find

$$|E(x, y)| \leq \frac{M}{3!}\left(|x - 1| + |y - 2|\right)^3 = \frac{7}{6}(3.2)^3$$

(b)

$$\begin{aligned}
\mathrm{d}V &= \frac{\partial V}{\partial r}\,\mathrm{d}r + \frac{\partial V}{\partial R}\,\mathrm{d}R + \frac{\partial V}{\partial h}\,\mathrm{d}h \\
&= \frac{\pi}{3}h\left(R + 2r\right)\mathrm{d}r + \frac{\pi}{3}h\left(2R + r\right)\mathrm{d}R + \frac{\pi}{3}\left(R^2 + Rr + r^2\right)\mathrm{d}h \\
&= \frac{\pi}{3}(3)\left[2 + 2(1)\right](0.01) + \frac{\pi}{3}(3)\left[2(2) + 1\right](0.01) + \frac{\pi}{3}\left[2^2 + (2)(1) + 1^2\right](0.03) \\
&= \frac{\pi}{100}\left(4 + 5 + 7\right) = \frac{16\pi}{100} = \frac{4\pi}{25}\ \text{in}^3
\end{aligned}$$

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