- This exam is worth 100 points and has 4 problems.
- **Show all work and simplify your answers!** Answers with no justification will receive no points unless otherwise noted.
- **Please begin each problem on a new page.**
- **DO NOT** leave the exam until you have satisfactorily scanned and uploaded your exam to Gradescope.
- You are taking this exam in a proctored and honor code enforced environment. **NO** calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5"$\times$ 11" crib sheet with writing on one side.
- Remote students are allowed use of a computer during the exam only for a live video of their hands and face and to view the exam in the Zoom meeting.

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0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2350/063023 (30 pts)] Penelope the platypus adventures about on the surface of a river, where distances are measured in meters. Suppose the temperature (°C) on the river's surface is given by
$$T(x,y) = -(2x^2-4)^2 + 4y^2$$
   (a) [10 pts] Penelope's current location is at $(1,3)$. Platypus risk overheating in Australian summers and can only cool down by immersing themselves in cold water.
      i. [5 pts] Find the direction Penelope should travel to reach lower temperatures the fastest.
      ii. [5 pts] What is the instantaneous rate of temperature change with respect to distance $(\mathrm{d}T/\mathrm{d}s)$ in this direction?
   (b) [20 pts] Penelope is now at the point $(2,-2)$, a location with a comfortable temperature.
      i. [6 pts] What is the equation of the level curve for this temperature? Sketch it.
      ii. [6 pts] What direction(s) should Penelope travel from this point to remain at this temperature? Report your answer(s) as unit vectors.
      iii. [8 pts] If Penelope were swimming along the path $\mathbf{r}(t) = (t+2)\,\mathbf{i} + \left(t^2+2t-2\right)\mathbf{j}$, what is the instantaneous rate of change of temperature with respect to time (seconds) $(\mathrm{d}T/\mathrm{d}t)$ when she reaches the point $(2,-2)$?

2. [2350/063023 (23 pts)] The following parts (a) and (b) are not related.
   (a) [15 pts] Consider
$$f(x,y) = \frac{y^2x^4 - 4x^4y + 4x^4}{y-2}$$
      i. [5 pts] What are the domain and range of $f(x,y)$?
      ii. [5 pts] Find $\lim\limits_{(x,y)\to(-2,2)} f(x,y)$ or show that it does not exist.
      iii. [5 pts] Is $f(x,y)$ continuous on $\mathbb{R}^2$? Justify your answer.
   (b) [8 pts] Find an equation of the tangent plane to the surface $xy+yz+xz=5$ at the point $(1,2,1)$. Write your answer in the form $ax+by+cz=d$.

3. [2350/063023 (31 pts)] You and your friend Chaplin are on a quest through a forest in search of a magical tome that can solve any math problem for you. A sorceress gave you a clue on its location: it is at the highest point on or within the boundary $3x^2+y^2=9$. The elevation in that region is given by
$$f(x,y) = -4(x-1)^2 - y^2 + 100$$
   (a) [5 pts] Chaplin questions whether there even is a highest point in this region. Is there any guarantee that you can offer him that there is a highest point in this region? (No calculations please, just write a sentence with any relevant facts/theorems that would convince Chaplin.)
   (b) [10 pts] Find and classify all critical points within the boundary.
   (c) [12 pts] Using Lagrange Multipliers, determine if there are any extrema on the boundary.
   (d) [4 pts] Based on the work done above, report the location of the magical item and the elevation at that point.

**CONTINUED ON REVERSE**

4. [2350/063023 (16 pts)] The following problems are not related.

   (a) [6 pts] Suppose you found the second order Taylor approximation, $T_2(x, y)$, centered at $(1, 2)$, of a function $g(x, y)$. You also know that throughout $\mathbb{R}^2$

   $$|g_{xx}| \leq 4 \qquad |g_{xy}| \leq e^2 \qquad |g_{yy|} \leq 1$$
   $$|g_{xxx}| \leq 3 \qquad |g_{xxy}| \leq 1 \qquad |g_{yyx}| \leq 7 \qquad |g_{yyy}| \leq 5$$

   You want to use $T_2(x, y)$ to estimate the value of $g(x, y)$ when $-3 \leq x - 1 \leq 3$ and $-0.2 \leq y - 2 \leq 0.2$. What is the maximum error you can expect the approximation to contain?

   (b) [10 pts] The volume of the frustum of a right circular cone is given by $V(r, R, h) = \frac{\pi}{3} h \left(R^2 + Rr + r^2\right)$, where $h$ is the frustum's height, $R$ is the radius of its base, and $r$ is the radius of its top. The measurements of the frustum are $r = 1, R = 2, h = 3$ inch with a possible error of $0.01$ inch in the radius measurements and $0.03$ inch in the height measurement. Use differentials to estimate the possible error in the computed volume.