1. [2350/061623 (45 pts)] Penelope the platypus leaves her burrow at $t = 0$ seconds and swims in the river in search of food, following the path given by (position in meters)

$$\mathbf{r}(t) = [\sin(t) - 2]\,\mathbf{i} + t\,\mathbf{j} + [\cos(t) - 1]\mathbf{k}$$

   (a) [8 pts] How fast is Penelope swimming?

   (b) i. [3 pts] What is Penelope's acceleration, $\mathbf{a}(t)$?

   ii. [4 pts] What are the tangential and normal components of Penelope's acceleration?

   iii. [3 pts] What is the unit normal to Penelope's path, $\mathbf{N}(t)$?

   (c) [5 pts] How much distance has Penelope traveled along the path when $t = \dfrac{5\pi}{3}$?

   (d) [10 pts] Penelope catches a crawfish at $t = \dfrac{5\pi}{3}$. She decides to take the crawfish to the surface of the river (described by the plane $z = 0$) to enjoy the snack. To get there, she swims in a straight line (tangent to the original path) at a speed of one meter per second. Find the arc length parameterization (think unit vector) of her straight line trajectory to the river's surface. Be sure to include a range for the parameter.

   (e) [6 pts] Using your answer for part (d), find where Penelope reaches the surface of the river ($z = 0$). How long did it take her to get there after catching the crawfish? Do not simplify your answer.

   (f) [6 pts] Using the position you found in part (e), how far is Penelope from her burrow? Do not simplify your answer.

SOLUTION:

(a) We seek Penelope's speed, $\|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\|$.

$$\mathbf{v}(t) = \mathbf{r}'(t) = \cos t\,\mathbf{i} + \mathbf{j} - \sin t\,\mathbf{k}$$
$$\|\mathbf{v}(t)\| = \sqrt{\cos^2 t + 1 + \sin^2 t} = \sqrt{2} \qquad \text{(constant)}$$

(b) i. The acceleration is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = -\sin t\,\mathbf{i} - \cos t\,\mathbf{k}$

ii. Since the speed is constant, there is no tangential acceleration and the acceleration consists of only the normal component.

$$a_T = \frac{\mathrm{d}\|\mathbf{v}\|}{\mathrm{d}t} = \frac{\mathrm{d}\sqrt{2}}{\mathrm{d}t} = 0$$

Alternatively, $$a_T = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|} = \frac{(\cos t\,\mathbf{i} + \mathbf{j} - \sin t\,\mathbf{k})\cdot(-\sin t\,\mathbf{i} - \cos t\,\mathbf{k})}{\|\cos t\,\mathbf{i} + \mathbf{j} - \sin t\,\mathbf{k}\|} = \frac{-\cos t\sin t + \sin t\cos t}{\sqrt{2}} = 0$$

$$a_N = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|(\cos t\,\mathbf{i} + \mathbf{j} - \sin t\,\mathbf{k})\times(-\sin t\,\mathbf{i} - \cos t\,\mathbf{k})\|}{\|\cos t\,\mathbf{i} + \mathbf{j} - \sin t\,\mathbf{k}\|} = \frac{\|-\cos t\,\mathbf{i} + \mathbf{j} + \sin t\,\mathbf{k}\|}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

Alternatively, $$\mathbf{a} = \mathbf{a}_T + \mathbf{a}_N = \mathbf{0} + \mathbf{a}_N \implies a_N = \|\mathbf{a}_N\| = \|\mathbf{a}\| = \|-\sin t\,\mathbf{i} - \cos t\,\mathbf{k}\| = 1$$

iii.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\cos t\,\mathbf{i} + \mathbf{j} - \sin t\,\mathbf{k}}{\|\cos t\,\mathbf{i} + \mathbf{j} - \sin t\,\mathbf{k}\|} = \frac{\cos t\,\mathbf{i} + \mathbf{j} - \sin t\,\mathbf{k}}{\sqrt{2}}$$
$$\mathbf{T}'(t) = \frac{-\sin t\,\mathbf{i} - \cos t\,\mathbf{k}}{\sqrt{2}} \implies \|\mathbf{T}'(t)\| = \frac{1}{\sqrt{2}}$$
$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = -\sin t\,\mathbf{i} - \cos t\,\mathbf{k}$$

Alternatively, since there is only a normal component of acceleration, $\mathbf{N}(t)$ must lie along $\mathbf{a}(t)$. $||\mathbf{a}(t)|| = 1$ so

$$\mathbf{N}(t) = \mathbf{a}(t) = -\sin(t)\mathbf{i} - \cos(t)\mathbf{k}$$

(c) We need the arclength from $t = 0$ to $t = \dfrac{5\pi}{3}$.

$$\int_0^{5\pi/3} \|\mathbf{r}'(t)\|\,\mathrm{d}t = \int_0^{5\pi/3} \sqrt{2}\,\mathrm{d}t = \frac{5\pi\sqrt{2}}{3} \text{ meters}$$

(d) We seek the tangent line at $t = \dfrac{5\pi}{3}$, requiring a point on the line and the line's direction. The direction of the tangent line is given by

$$\mathbf{v}\left(\frac{5\pi}{3}\right) = \left\langle \frac{1}{2}, 1, \frac{\sqrt{3}}{2} \right\rangle$$

and a point on the line is

$$\mathbf{r}\left(\frac{5\pi}{3}\right) = \frac{-\sqrt{3}-4}{2}\mathbf{i} + \frac{5\pi}{3}\mathbf{j} - \frac{1}{2}\mathbf{k}$$

The arc length parameterization of the tangent line, $\mathbf{L}$, is:

$$\mathbf{L}(s) = \mathbf{r}\left(\frac{5\pi}{3}\right) + s\left[\frac{\mathbf{v}\left(\frac{5\pi}{3}\right)}{\left\|\mathbf{v}\left(\frac{5\pi}{3}\right)\right\|}\right] = \left\langle \frac{-\sqrt{3}-4}{2}, \frac{5\pi}{3}, -\frac{1}{2} \right\rangle + \frac{s}{\sqrt{2}}\left\langle \frac{1}{2}, 1, \frac{\sqrt{3}}{2} \right\rangle, \ s \geq 0$$

(e)

$$z = 0 \implies \frac{\sqrt{3}}{2\sqrt{2}}s - \frac{1}{2} = 0 \implies s = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$$

So the location is $\mathbf{L}\left(\dfrac{\sqrt{2}}{\sqrt{3}}\right) = \left(\dfrac{1}{2\sqrt{3}} - \dfrac{4+\sqrt{3}}{2}\right)\mathbf{i} + \left(\dfrac{5\pi}{3} + \dfrac{1}{\sqrt{3}}\right)\mathbf{j} + 0\mathbf{k}$. The time it takes to get there is $\dfrac{\sqrt{2}}{\sqrt{3}} = \dfrac{\sqrt{6}}{3}$ seconds.

(f) Use the distance formula. Penelope's burrow is at the point $(-2, 0, 0)$ so the distance is

$$\mathbf{r}_2\left(1/\sqrt{3}\right) - \mathbf{r}(0) = \sqrt{\left(\frac{1}{2\sqrt{3}} - \frac{4+\sqrt{3}}{2} + 2\right)^2 + \left(\frac{5\pi}{3} + \frac{1}{\sqrt{3}} - 0\right)^2 + (0-0)^2} \qquad \text{meters from her burrow}$$

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2. [2350/061623 (12 pts)] Consider the quadric surface

$$y^2 - 6y + z^2 - x^2 + 2x = -4$$

(a) [7 pts] Classify the quadric surface, describe how it is oriented and where it is in space.

(b) [5 pts] Sketch and describe the trace when $x = 1$. Label your sketch appropriately.

SOLUTION:

(a) Completing the square results in

$$y^2 - 6y + 9 - 9 + z^2 - (x^2 - 2x + 1 - 1) = -4$$

$$(y-3)^2 + z^2 - (x-1)^2 = 4$$

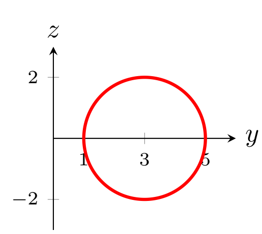
$$-\frac{(x-1)^2}{4} + \frac{(y-3)^2}{4} + \frac{z^2}{4} = 1$$

This is a hyperboloid of one sheet. Its axis is along the line with parametric equations $x = 1 + t, y = 3, z = 0$ and it is shifted so that its center is at $(1, 3, 0)$.

(b) When $x = 1$ we have

$$\frac{(y-3)^2}{4} + \frac{z^2}{4} = 1 \implies (y-3)^2 + z^2 = 4$$

The trace is a circle of radius 2, centered at $y = 3, z = 0$ in the plane $x = 1$.



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3. [2350/061623 (15 pts)] Suppose the acceleration experienced by a particle moving along a path in space is $\mathbf{a}(t) = \langle 2, 0, 24t^2 \rangle$. Furthermore, suppose that the particle's velocity at time $t = 0$ is $\mathbf{j}$ and that the particle is at the point $(3, 7, 2)$ when $t = -1$.

(a) [10 pts] Where is the particle when $t = 1$?

(b) [5 pts] Is the particle's acceleration vector ever orthogonal to it velocity vector? If so, when? If not, justify why not.

SOLUTION:

(a)

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,\mathrm{d}t = \left\langle \int 2\,\mathrm{d}t, \int 0\,\mathrm{d}t, \int 24t^2\,\mathrm{d}t \right\rangle = \left\langle 2t + c_1, c_2, 8t^3 + c_3 \right\rangle$$

$$\mathbf{v}(0) = \mathbf{j} = \langle 0, 1, 0\rangle = \langle c_1, c_2, c_3\rangle \implies c_1 = c_3 = 0, c_2 = 1$$

$$\mathbf{v}(t) = \left\langle 2t, 1, 8t^3 \right\rangle$$

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,\mathrm{d}t = \left\langle \int 2t\,\mathrm{d}t, \int 1\,\mathrm{d}t, \int 8t^3\,\mathrm{d}t \right\rangle = \left\langle t^2 + c_1, t + c_2, 2t^4 + c_3 \right\rangle$$

$$\mathbf{r}(-1) = \langle 3, 7, 2\rangle = \left\langle (-1)^2 + c_1, -1 + c_2, 2(-1)^4 + c_3 \right\rangle \implies c_1 = 2, c_2 = 8, c_3 = 0$$

$$\mathbf{r}(t) = \left\langle t^2 + 2, t + 8, 2t^4 \right\rangle$$

At $t = 1$, the particle is at $\mathbf{r}(1) = \left\langle 1^2 + 2, 1 + 8, 2(1)^4 \right\rangle = \langle 3, 9, 2\rangle$.

(b) We know $\mathbf{a}(t)$ and $\mathbf{v}(t)$ are orthogonal when

$$\mathbf{a}(t) \cdot \mathbf{v}(t) = \langle 2, 0, 24t^2\rangle \cdot \langle 2t, 1, 8t^3\rangle = 4t + 8(24)t^5 = 0$$

$$4t(1 + 48t^4) = 0 \implies t = 0 \qquad \text{(since } 1 + 48t^4 > 0 \text{ for all t)}$$

Thus, the particle's acceleration and velocity vectors are orthogonal at $t = 0$.

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4. [2350/061623 (20 pts)] Consider the lines

$$L_1 : \frac{x+3}{4} = \frac{z+1}{2},\ y = 4 \quad \text{and} \quad L_2 : \frac{x-1}{2} = y - 4 = \frac{z-1}{2}$$

(a) [10 pts] Show that the lines are neither parallel nor skew. Justify your answer.

(b) [10 pts] Find an equation of the plane containing these lines. Write your answer in standard form $ax + by + cz = d$.

SOLUTION:

Begin by writing the equations in parametric form:

$$L_1 : x(u) = 4u - 3,\ y(u) = 4,\ z(u) = 2u - 1 \quad \text{and} \quad L_2 : x(w) = 2w + 1,\ y(w) = w + 4,\ z(w) = 2w + 1$$

(a) The direction vector for L_1 is $\mathbf{v}_1 = \langle 4, 0, 2\rangle$ and for L_2 is $\mathbf{v}_2 = \langle 2, 1, 2\rangle$. These vectors are not scalar multiples of one another, therefore the lines are not parallel. To see if the lines intersect (and thus are not skew), we see if u and w exist such that

$$4u - 3 = 2w + 1 \tag{1}$$

$$4 = w + 4 \tag{2}$$

$$2u - 1 = 2w + 1 \tag{3}$$

Eq. (2) requires $w = 0$. Then Eq. (1) gives $u = 1$ which also satisfies Eq. (3). Using either $u = 1$ in the parametric equations for L_1 or $w = 0$ in the equations for L_2 gives the point of intersection as $(1, 4, 1)$. Since the lines intersect, they are not skew. Note that it is sufficient to only show that they intersect to prove that they are not parallel or skew.

(b) To find the equation of a plane, we need a point in the plane, one on both lines in this case, $(1, 4, 1)$, and a vector normal to the plane. Since the plane must contain both lines, the cross product of their direction vectors will give a normal.

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & 2 \\ 2 & 1 & 2 \end{vmatrix} = -2\,\mathbf{i} - 4\,\mathbf{j} + 4\,\mathbf{k}$$

Then the plane's equation is

$$-2(x-1) - 4(y-4) + 4(z-1) = 0 \quad \text{or in standard form} \quad -2x - 4y + 4z = -14 \quad \text{or} \quad x + 2y - 2z = 7$$

5. [2350/061623 (8 pts)] $100\sqrt{3}$ ft-lb of work are done in moving a crate 10 feet along a horizontal boat dock. The constant force, $\mathbf{F}$, used to move the crate is applied at an angle of $30°$ above the horizontal. If this same force produces a torque of magnitude 5 ft-lb when applied at the same angle to the end of a wrench to loosen a bolt, how long is the wrench?

SOLUTION:

$$\text{Work} = \mathbf{F} \cdot \mathbf{D} = \|\mathbf{F}\|\|\mathbf{D}\|\cos\theta$$

$$100\sqrt{3} = \|\mathbf{F}\|(10)\cos 30°$$

$$100\sqrt{3} = \|\mathbf{F}\|(10)\left(\frac{\sqrt{3}}{2}\right)$$

$$\|\mathbf{F}\| = 20 \text{ lb}$$

Then

$$\|\boldsymbol{\tau}\| = \|\mathbf{r} \times \mathbf{F}\| = \|\mathbf{r}\|\|\mathbf{F}\|\sin\theta$$

$$5 = \|\mathbf{r}\|(20)\sin 30°$$

$$5 = \|\mathbf{r}\|(20)\left(\frac{1}{2}\right)$$

$$\|\mathbf{r}\| = 0.5 \text{ ft} = 6 \text{ in}$$