1. (32 pts) Suppose the density of the surface $z = 1 - x^2$ is $\delta = |x|y$ g/cm^2 and consider the vector field

$$\mathbf{F} = \langle 3x + \cos y, 2y + \sin z, e^x + 5z \rangle$$

(a) Find the mass of the part of the surface lying above the region in the xy-plane between $y = 0$ and $y = 2$.

(b) Find the outward flux of $\mathbf{F}$ through the **closed** surface enclosing the region below $z = 1 - x^2$, above the xy-plane and between $y = 0$ and $y = 2$.

Solution:

(a)

$$g(x, y, z) = x^2 + z \implies \nabla g = \langle 2x, 0, 1 \rangle \implies \|\nabla g\| = \sqrt{4x^2 + 1}$$

Project surface onto the xy-plane gives $\mathbf{p} = \mathbf{k}$, integration region $-1 \le x \le 1, 0 \le y \le 2$ and $|\nabla g \cdot \mathbf{p}| = 1$

$$\begin{aligned}
\text{Mass} &= \int_{-1}^{1} \int_{0}^{2} |x| y \sqrt{4x^2 + 1} \, \mathrm{d}y \, \mathrm{d}x \quad \text{(integrand even in } x \text{ and separable)} \\
&= 2 \left(\int_{0}^{1} x\sqrt{4x^2 + 1} \, \mathrm{d}x \right) \left(\int_{0}^{2} y \, \mathrm{d}y \right) \\
&= \frac{1}{4} \left(\int_{1}^{5} u^{1/2} \, \mathrm{d}u \right) \left(\frac{y^2}{2} \Big|_{0}^{2} \right) = \frac{1}{3} \left(5\sqrt{5} - 1 \right)
\end{aligned}$$

(b) The surface $\mathcal{S}$ and the region $\mathcal{W}$ it encloses satisfy the hypotheses of Gauss' (Divergence) Theorem with

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{\partial}{\partial x}(3x + \cos y) + \frac{\partial}{\partial y}(2y + \sin z) + \frac{\partial}{\partial z}(e^x + 5z) = 3 + 2 + 5 = 10$$

$$\begin{aligned}
\text{Flux} &= \iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_{\mathcal{W}} \boldsymbol{\nabla} \cdot \mathbf{F} \, \mathrm{d}V = \int_{-1}^{1} \int_{0}^{2} \int_{0}^{1 - x^2} 10 \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x \\
&= 10 \int_{-1}^{1} \int_{0}^{2} \left(1 - x^2 \right) \mathrm{d}y \, \mathrm{d}x = 20 \int_{-1}^{1} \left(1 - x^2 \right) \mathrm{d}x = 40 \int_{0}^{1} \left(1 - x^2 \right) \mathrm{d}x = \frac{80}{3}
\end{aligned}$$

2. (16 pts) Find the area under the graph of $z = 100(x^2 + 2y^2)$ lying above the second quadrant portion of the **curve** $x^2 + y^2 = 4$.

Solution: The area is given by $\displaystyle\int_{\mathcal{C}} f(x, y) \, \mathrm{d}s$ where $f(x, y) = 100(x^2 + 2y^2)$. $\mathcal{C}$ can be parameterized by

$$\mathbf{r}(t) = 2\cos t \, \mathbf{i} + 2\sin t \, \mathbf{j}, \; \pi/2 \le t \le \pi \implies \mathbf{r}'(t) = -2\sin t \, \mathbf{i} + 2\cos t \, \mathbf{j} \implies \|\mathbf{r}'(t)\| = 2$$

Thus

$$\begin{aligned}
\text{Area} &= \int_{\mathcal{C}} 100(x^2 + 2y^2) \, \mathrm{d}s = \int_{\pi/2}^{\pi} 100 \left(4\cos^2 t + 8\sin^2 t \right) (2) \, \mathrm{d}t = 800 \int_{\pi/2}^{\pi} \left(\cos^2 t + 2\sin^2 t \right) \mathrm{d}t \\
&= 800 \int_{\pi/2}^{\pi} (1 + \sin^2 t) \, \mathrm{d}t = 800 \int_{\pi/2}^{\pi} \left(\frac{3}{2} - \frac{1}{2}\cos 2t \right) \mathrm{d}t = 400 \left(3t + \frac{1}{2}\sin 2t \right) \Bigg|_{\pi/2}^{\pi} = 600\pi
\end{aligned}$$

3. (16 pts) I am doing laps around the unit circle (counterclockwise) in the presence of the force field

$$\mathbf{F} = \left\langle Axy - By^3, 4y + 3x^2 - 3xy^2 \right\rangle$$

(a) After having gone from $(1,0)$ to $(0,1)$, I am already getting tired from all of the work I've done. A friend standing nearby tells me to chill because when I get back to $(1,0)$ I will have done no work at all. What are A and B? Briefly explain.

(b) If I go around the circle too much, I'll get dizzy so my friend tells me to go from $(-1,0)$ to $(3,-2)$ along the path $y = \sqrt{x+1}(x-2)^{300}(x-4)^{301}$ instead. How much work will I do walking on that path?

Solution:

(a) The fact that no work is done when traversing a closed path implies that the vector field is conservative so that

$$\frac{\partial}{\partial x}\left(4y + 3x^2 - 3xy^2\right) = \frac{\partial}{\partial y}\left(Axy - By^3\right) \implies 6x - 3y^2 = Ax - 3By^2 \implies A = 6, B = 1$$

(b) Parameterizing the given path would not be a pleasant experience but that is not necessary. There are two ways to handle this. First, since the vector field is conservative, line integrals are path independent so we could pick another path between the given points (perhaps a line segment). This may still be too much work (no pun intended). The other approach is to find a potential function and use the fundamental theorem for line integrals to compute the work. To that end,

$$\frac{\partial f}{\partial x} = 6xy - y^3 \implies f(x,y) = \int \left(6xy - y^3\right) \mathrm{d}x = 3x^2y - xy^3 + g(y)$$

Then

$$\frac{\partial f}{\partial y} = 3x^2 - 3xy^2 + \frac{\mathrm{d}g}{\mathrm{d}y} = 4y + 3x^2 - 3xy^2 \implies \frac{\mathrm{d}g}{\mathrm{d}y} = 4y \implies g(y) = \int 4y\,\mathrm{d}y \implies g(y) = 2y^2 + c$$

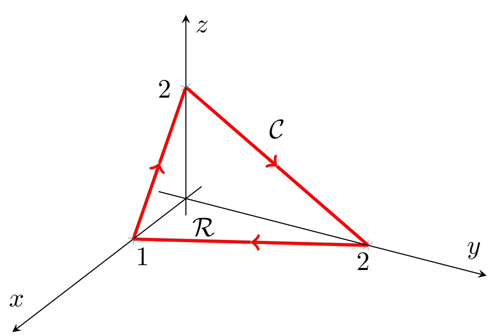
So the potential function for $\mathbf{F}$ is $f(x,y) = 3x^2y - xy^3 + 2y^2 + c$ and the work is then

$$\int_{(-1,0)}^{(3,-2)} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = f(3,-2) - f(-1,0) = 3(9)(-2) - 3(-8) + 2(4) + c - (0+c) = -22$$

4. (20 pts) Find the circulation of $\mathbf{V}(x,y,z) = \mathbf{i} + (x+yz)\,\mathbf{j} + \left(xy - \cos^2\sqrt{z}\right)\mathbf{k}$ around the closed path consisting of the straight line segments connecting the points $(1,0,0)$, $(0,0,2)$, $(0,2,0)$ and $(1,0,0)$, in that order, by completely setting up an appropriate **surface** integral. Do not evaluate your integral, but your final answer should include a fully simplified integrand, correct bounds, *etc.*

Solution:

The path is shown in the following figure.



We will use Stokes' Theorem to find the circulation. The surface over which we will integrate is given by the plane in which the path lies, namely,

$$\frac{x}{1} + \frac{y}{2} + \frac{z}{2} = 1 \implies 2x + y + z = 2 \implies g(x,y,z) = 2x + y + z \text{ and } \nabla g = 2\,\mathbf{i} + \mathbf{j} + \mathbf{k}$$

(This was easily obtained since we knew where the plane intersects the coordinate axes. A point and two vectors in the plane could also have been used to find the plane's equation).

To obtain the orientation of the surface induced by the orientation of its boundary requires the use of $-\nabla g$. Projecting the surface onto the xy-plane gives $\mathbf{p} = \mathbf{k}$ and $|\nabla g \cdot \mathbf{p}| = 1$ with the area of integration

$$\mathcal{R} = \left\{(x, y,) \in \mathbb{R}^2 \,\middle|\, 0 \le x \le 1, 0 \le y \le 2 - 2x\right\}$$

Note that the surface could have been projected onto the xz- or yz-plane.

We need the curl of $\mathbf{V}$, given as

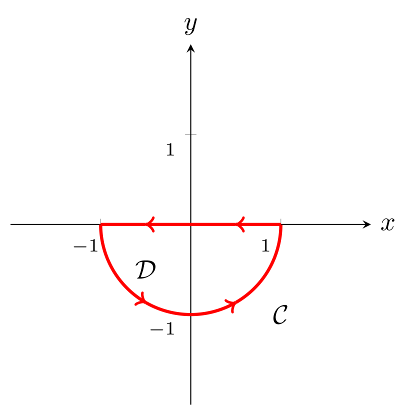
$$\boldsymbol{\nabla}\times\mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & x + yz & xy - \cos^2\sqrt{z} \end{vmatrix} = (x - y)\,\mathbf{i} - y\,\mathbf{j} + \mathbf{k}$$

Then

$$\begin{aligned} \text{Circulation} = \oint_{\mathcal{C}} \mathbf{V} \cdot \mathrm{d}\mathbf{r} = \iint_{\mathcal{S}} \boldsymbol{\nabla}\times\mathbf{V} \cdot \mathrm{d}\mathbf{S} &= \iint_{\mathcal{R}} (\boldsymbol{\nabla}\times\mathbf{V}) \cdot \frac{-\nabla g}{|\nabla g \cdot \mathbf{p}|}\,\mathrm{d}A \\ &= \iint_{\mathcal{R}} [(x - y)\,\mathbf{i} - y\,\mathbf{j} + \mathbf{k}] \cdot \frac{(-2\,\mathbf{i} - \mathbf{j} - \mathbf{k})}{1}\mathrm{d}A \\ &= \int_0^1 \int_0^{-2x+2} (-2x + 3y - 1)\,\mathrm{d}y\,\mathrm{d}x \end{aligned}$$

5. (16 pts) Use Green's Theorem to evaluate $\oint_{\mathcal{C}} -y^3\,\mathrm{d}x + \left(x^3 + \sqrt{y^3 + 1}\right)\mathrm{d}y$. $\mathcal{C}$ is the closed counterclockwise path consisting of the bottom half of the unit circle and the portion of the x-axis with $-1 \le x \le 1$.

Solution: The path and region are shown in the following figure.



$$P(x, y) = -y^3 \implies \frac{\partial P}{\partial y} = -3y^2 \quad \text{and} \quad Q(x, y) = x^3 + \sqrt{y^3 + 1} \implies \frac{\partial Q}{\partial x} = 3x^2$$

Thus

$$\oint_{\mathcal{C}} -y^3\,\mathrm{d}x + \left(x^3 + \sqrt{y^3 + 1}\right)\,\mathrm{d}y = \iint_{\mathcal{D}} \left(3x^2 + 3y^2\right)\mathrm{d}A \overset{\text{polar}}{\underset{\text{coords}}{=}} 3\int_{\pi}^{2\pi}\int_0^1 r^3\,\mathrm{d}r\,\mathrm{d}\theta = \frac{3\pi}{4}$$

Note that this could also be considered using the flux/divergence/normal form of Green's Theorem with

$$P(x, y) = x^3 + \sqrt{y^3 + 1} \;\text{ and }\; Q(x, y) = y^3$$

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END OF EXAM