This exam has 5 problems. Show all your work and simplify your answers. Answers with missing or insufficient justification will receive no points. You are allowed one 8.5×11-in page of notes (ONE side). You may NOT use a calculator, smartphone, smartwatch, the Internet or any other electronic device.

1. (18 pts) Evaluate $\displaystyle\int_0^1 \int_{\sqrt[3]{z}}^1 \int_0^{\ln 3} \frac{\pi e^{2x}\sin(\pi y^2)}{y^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

   Solution:

$$\int_0^1 \int_{\sqrt[3]{z}}^1 \int_0^{\ln 3} \frac{\pi e^{2x}\sin\left(\pi y^2\right)}{y^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^1 \int_{\sqrt[3]{z}}^1 \frac{\pi \sin(\pi y^2)}{y^2}\left(\frac{1}{2}e^{2x}\bigg|_0^{\ln 3}\right)\mathrm{d}y\,\mathrm{d}z$$

$$= 4\pi \int_0^1 \int_{\sqrt[3]{z}}^1 \frac{\sin\left(\pi y^2\right)}{y^2}\mathrm{d}y\,\mathrm{d}z$$

$$\underset{\text{order}}{\overset{\text{switch}}{=}} 4\pi \int_0^1 \int_0^{y^3} \frac{\sin\left(\pi y^2\right)}{y^2}\mathrm{d}z\,\mathrm{d}y$$

$$= 4\pi \int_0^1 y \sin\left(\pi y^2\right)\,\mathrm{d}y$$

$$\overset{u=\pi y^2}{=} 4\pi \int_0^{\pi} \frac{1}{2\pi}\sin u\,\mathrm{d}u = 4$$

2. (20 pts) Consider the solid between two concentric spheres of radii r_1 and r_2 with $0 < r_1 < r_2$. The mass density of the solid varies inversely with the cube of the distance from the center of the spheres, that is, mass density = k/distance^3. Find the constant of proportionality, k, in terms of the other variables, if the total mass of the solid is M.

   Solution: Use spherical coordinates, placing the origin at the center of the concentric spheres. Then the density can be written as $\delta(\rho,\phi,\theta) = \dfrac{k}{\rho^3}$.

$$\text{Total Mass} = M = \iiint \delta\,\mathrm{d}V = \int_0^{2\pi}\int_0^{\pi}\int_{r_1}^{r_2} \frac{k}{\rho^3}\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

$$= \left(\int_0^{2\pi}\mathrm{d}\theta\right)\left(\int_0^{\pi}\sin\phi\,\mathrm{d}\phi\right)\left(\int_{r_1}^{r_2}\frac{k}{\rho}\,\mathrm{d}\rho\right)$$

$$= 2\pi\left(\cos\phi\bigg|_{\pi}^{0}\right)\left(k\ln\rho\bigg|_{r_1}^{r_2}\right) = 4\pi k \ln\frac{r_2}{r_1}$$

$$\implies k = \frac{M}{4\pi\ln(r_2/r_1)}$$

3. (14 pts) Let Q be the solid region contained inside both the cylinder $x^2+(y-1)^2 = 1$ and the sphere $x^2+y^2+z^2 = 4$. Suppose the temperature at the point (x,y,z) in Q is given by

$$T(x,y,z) = \frac{4-z}{\sqrt{1+x^2+y^2}}$$

   Set up, but **do not evaluate** integral(s) to find the average temperature in Q using cylindrical coordinates.

   Solution: The integrand becomes $\dfrac{4-z}{\sqrt{1+x^2+y^2}} = \dfrac{4-z}{\sqrt{1+r^2}}$. The equation of the sphere in cylindrical coordinates becomes $r^2+z^2 = 4$ and that of the cylinder is $r = 2\sin\theta$. The region of integration is described by the inequalities $-\sqrt{4-r^2} \le z \le \sqrt{4-r^2}$, $0 \le r \le 2\sin\theta$, $0 \le \theta \le \pi$. Thus

$$T_{\text{avg}} = \frac{\displaystyle\int_0^{\pi}\int_0^{2\sin\theta}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} \frac{(4-z)r}{\sqrt{1+r^2}}\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta}{\displaystyle\int_0^{\pi}\int_0^{2\sin\theta}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta}$$

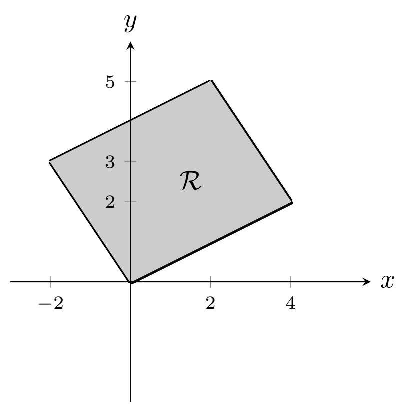
4. (22 pts) Use a change of variables to find the volume of the solid region lying below the surface

$$z = (3x+2y)(2y-x)^{3/2}$$

and above the plane region $\mathcal{R}$ bounded by the parallelogram with vertices $(0,0)$, $(-2,3)$, $(2,5)$, $(4,2)$.

Solution: To find the requested volume, we need to compute $\iint_{\mathcal{R}} (3x+2y)(2y-x)^{3/2}\,\mathrm{d}A$, suggesting the change of variables $u = 3x+2y$ and $v = 2y-x$.

Using the vertices of the region $\mathcal{R}$ depicted below, the equation of the lines with positive slope right are $2y-x=8$ (top) and $2y-x=0$ (bottom) and those with negative slope are $3x+2y=0$ (left) and $3x+2y=16$ (right). Thus, $\mathcal{R}$ is given by $0 \le 3x+2y \le 16$ and $0 \le 2y-x \le 8$, which gives the new region of integration as $0 \le u \le 16$ and $0 \le v \le 8$.



Now $u - v = 4x \Rightarrow x = \frac{1}{4}(u-v)$ so that $y = \frac{1}{2}(v+x) = \frac{1}{8}(u+3v)$ and

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} \end{vmatrix} = \frac{3}{32} - \left(-\frac{1}{32}\right) = \frac{1}{8} \quad \text{and} \quad f(u,v) = uv^{3/2}$$

Thus

$$\text{Volume} = \iint_{\mathcal{R}} (3x+2y)(2y-x)^{3/2}\,\mathrm{d}A = \int_0^8\int_0^{16} uv^{3/2}\left|\frac{1}{8}\right|\mathrm{d}u\,\mathrm{d}v = \frac{1}{8}\int_0^8 \frac{1}{2}u^2\Big|_0^{16} v^{3/2}\,\mathrm{d}v$$

$$= \frac{1}{(8)(2)}4^4\int_0^8 v^{3/2}\,\mathrm{d}v = 16\left(\frac{2}{5}\right)v^{5/2}\Big|_0^8 = \frac{32}{5}(64)(2\sqrt{2}) = \frac{4096\sqrt{2}}{5}$$

5. (26 pts) Let $\mathcal{W}$ be the solid 3D object that satisfies the following inequalities:

$$-R \le x \le 0, \quad \frac{H}{R}\sqrt{x^2+y^2} \le z \le H, \quad 0 \le y \le \sqrt{R^2-x^2}$$

where H and R are positive constants.

(a) Sketch and shade a cross section of the object in the rz-plane (*i.e.*, for any constant θ through which the object passes).

(b) Sketch and shade the projection of the object onto the xy-plane.

(c) Each of the following integrals will be used to compute the volume of $\mathcal{W}$. Copy the following integrals onto your paper and provide the six (6) proper limits for each one, using the given order of integration. Your limits must correspond to the region as given, not an equivalent one in a different portion of $\mathbb{R}^3$. **Do NOT evaluate** any of the integrals.

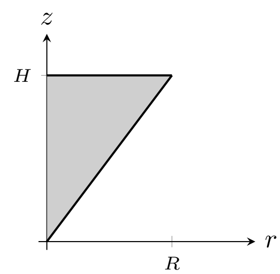
i. Volume($\mathcal{W}$) = $\int \int \int \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}z$

ii. Volume($\mathcal{W}$) = $\int \int \int r\, \mathrm{d}r\, \mathrm{d}z\, \mathrm{d}\theta$

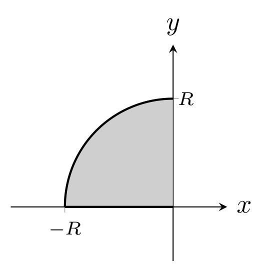
iii. Volume($\mathcal{W}$) = $\int \int \int \rho^2 \sin\phi\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\theta$

Solution:

(a) Sketch.



(b) Sketch.



(c) $\mathcal{W}$ is the "inside" of the cone of radius R and height H with vertex at the origin residing above Quadrant II.

i. $z = \frac{H}{R}\sqrt{x^2+y^2} \implies z^2 = \frac{H^2}{R^2}\left(x^2+y^2\right) \implies y^2 = \left(\frac{Rz}{H}\right)^2 - x^2 \implies y = \pm\sqrt{\left(\frac{Rz}{H}\right)^2 - x^2}$

Arbitrary x, z: y enters the region at 0 and exits at $\sqrt{\left(\frac{Rz}{H}\right)^2 - x^2}$; project onto xz-plane

Arbitrary z: x enters the projected region at $-\frac{Rz}{H}$ and exits at 0

Sum z from 0 to H

$$\int_0^H \int_{-Rz/H}^0 \int_0^{\sqrt{(Rz/H)^2 - x^2}} \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}z$$

ii. $r = \frac{R}{H}z$

Arbitrary θ, z: r enters the region at 0 and exits at $\frac{R}{H}z$

Sum z from 0 to H

Sum θ from $\pi/2$ to π

$$\int_{\pi/2}^{\pi} \int_0^H \int_0^{Rz/H} r \, \mathrm{d}r \, \mathrm{d}z \, \mathrm{d}\theta$$

iii. Top of the cone is the plane $\rho = H \sec\phi$.

Arbitrary ϕ, θ: ρ enters at 0 and exits at $H \sec\phi$.

Sum ϕ from 0 to $\tan^{-1} \frac{R}{H}$

Sum θ from $\pi/2$ to π

$$\int_{\pi/2}^{\pi} \int_0^{\tan^{-1} \frac{R}{H}} \int_0^{H \sec\phi} \rho^2 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta$$

END OF EXAM
